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Binary Fluids with Long Range Segregating Interaction. I: Derivation of Kinetic and Hydrodynamic Equations

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We study the evolution of a two component fluid consisting of "blue" and "red" particles which interact via strong short range (hard core) and weak long range pair potentials. At low temperatures the equilibrium state of the system is one in which there are two coexisting phases. Under suitable choices of space-time scalings and system parameters we first obtain (formally) a mesoscopic kinetic Vlasov–Boltzmann equation for the one particle position and velocity distribution functions, appropriate for a description of the phase segregation kinetics in this system. Further scalings then yield Vlasov–Euler and incompressible Vlasov–Navier–Stokes equations. We also obtain, via the usual truncation of the Chapman–Enskog expansion, compressible Vlasov–Navier–Stokes equations.

KEY WORDS: Binary fluids; phase segregation; kinetic and hydrodynamic equations; long-range interactions.

1. INTRODUCTION

The process of phase segregation in which a system evolves from an initial unstable homogeneous state into a final equilibrium state consisting of two coexisting phases is of continuing theoretical and practical interest [GSS, FLP, L]. Such a process occurs whenever the system, which is initially at values of the thermodynamic parameters, say temperature T_0 and pressure p_0, corresponding to a single homogeneous phase has its parameters "suddenly"

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changed to new values, say T and p, at which there is a coexistence of phases.

This happens, for example, when an alloy is "quenched" from a high temperature melt or solid to a low temperature solid state by sudden cooling [GSS]. After such a quench the system finds itself in an unstable (or metastable) situation, as far as the spatial concentrations, which have not been able to adjust rapidly enough to the "sudden" quench, are concerned and domains of the two equilibrium phases form and start growing in time. This proceeds until there are "in the final state" only two regions of pure equilibrium phases separated by an interface. Since the kinetics of the domain growth have a profound influence on the properties of the alloy, this problem has been and continues to be extensively studied both theoretically and experimentally [GSS]. For such alloy systems the segregation process takes place mainly through the (anti)diffusion of the two components—from a uniformly mixed state to a demixed one. There are no macroscopic matter or energy flows since the system is a solid and has a high heat conductivity which keeps the temperature equal to some constant ambient value. The only relevant conserved quantities are therefore the particle numbers of the two components and the macroscopic equations describing the process are fairly well established: these are the well known Cahn–Hilliard equations [CH] and variations on them. We refer the reader to reviews on this subject [GSS].

The situation is much less clear for phase segregation in fluids where macroscopic flows of matter and heat are important. There are now additional conservation laws for momentum and energy and there is no general consensus even on what hydrodynamical equations are most appropriate for describing the macroscopic evolution of the system [S, OP, AGA]. In particular, it is not clear which is the correct coupling between the Cahn–Hilliard equation for the order parameter and the Navier–Stokes equation for the fluid velocity.

To make a start on the mathematical analysis of such processes we investigate a model binary fluid introduced in [BL] where the process of phase segregation was studied numerically.

In the present work we derive general equations appropriate both in the one phase and in the coexistence region. In part II we consider applications to the segregation process including an analysis of new numerical results. Many of our discussions here will be semi-heuristic. In particular, we will not go into detail about the domain of validity of the technical conditions necessary for the rigorous mathematical establishment of the results.

The model we study is composed of two types of particles, call them red and blue. There are N_r red and N_b blue particles in a cubic box of volume $\Lambda = L^d$; we will generally consider $d = 3$ and use periodic boundary conditions. The particles all have unit mass and hard core diameter a. Particles of different kind also interact with each other through a long range pair potential of the Kac type, having a range ℓ and a strength A_ℓ. By properly choosing A_ℓ, we obtain, in the limit $\ell \to \infty$, a system whose equilibrium properties are described by a mean-field type phase diagram exhibiting a demixing phase transition for temperature $T < T_c$ [LP].

This transition is essentially independent of the hard core size a and the dimensionless microscopic particle densities $\rho_r a^3$ and $\rho_b a^3$ can therefore be arbitrarily small in the demixed phases. This means that we can have a situation in which, at least in principle, the whole phase transition is well described by a Vlasov–Boltzmann type of kinetic equation. We will in fact see that we can, by suitably scaling space and time and the densities, obtain, at least on the formal level, a set of nonlinear Vlasov–Boltzmann (VB) equations, describing the evolution of the one particle distribution functions $f^\alpha(q, v, t)$, $\alpha = r, b$.

The VB equations we derive are of a form similar to ones conjectured for a one component fluid with hard cores and an attractive long range interaction [DS, G]. Such a system however requires the hard cores for stabilization against collapse [LP] and ρa^3 is greater than 1/3 in the liquid phase. It is therefore not clear that a VB equation is an appropriate kinetic description of such a liquid–vapor transition. This is the motivation for introducing the binary model we consider here.

We discuss the scalings necessary to go from a microscopic Hamiltonian description of the time evolution to the VB equations in Section 2 leaving a formal derivation, in the spirit of Cercignani [C] and Lanford [Lan], to Appendix C. The equations themselves are of the same form as those used in [BL] for the kinetics in the coexistence regime of this system. Their numerical results for the time evolution and the analysis of the stationary states showed that these VB equations for the one particle distributions f^α indeed lead to the phase segregated state expected from purely equilibrium considerations.

While the mesoscopic description in terms of the one-particle distribution functions is a great simplification compared to the full microscopic representation, it is still more complicated than the macroscopic theory that treats the binary system as a continuum with well defined local density $\rho(x, t)$, concentration difference $\varphi(x, t)$, velocity $u(x, t)$ and temperature $T(x, t)$. The derivation of hydrodynamic equations from the Boltzmann equation (which one expects to be structurally of the same form as those describing dense binary fluids) is closely related to the problem of finding approximate solutions of the Boltzmann equation. The reason for this is that the fluid dynamic variables are defined and change on space and time scales which are very large when measured in units of the mean free path and mean free time between collisions, i.e., the kinetic or mesoscopic scale. Therefore, it can be expected that the system will reach a state close to local equilibrium in a macroscopically very small time interval, meaning that $f^{\alpha}(x, v, t)$ should stay close to local Maxwellians, with parameters ρ^{α}, u and T, which change slowly on the kinetic scale. The big disparity between the kinetic and hydrodynamic scales suggests looking for a solution of the Boltzmann equation as a series expansion in the scale parameter which is the ratio of these two scales. Many rigorous results in this direction have been obtained in recent years, especially for the Euler (E) and the incompressible Navier–Stokes (INS) equations. The situation is less satisfactory in the case of the compressible Navier–Stokes (NS). This is a consequence of the fact that while the E and INS equations correspond to well defined scaling limits, in which the mean free path goes to zero, there is no such scaling limit for the NS equations as can be seen from the fact that these equations are not invariant under scaling [DEL].

Having obtained the VB equations we turn to the derivation of hydrodynamic equations. The results available for these equations are fewer than for the Boltzmann equation. In Section 3 we present a rigorous derivation of the Vlasov–Euler (VE) equations for this system, which differs from the usual Euler equations by the presence of self-consistent forces coming from the Vlasov terms. We do this by adapting to this case the method of Caflisch [Ca80], i.e., we prove that the Hilbert expansion is asymptotic, by showing that the remainder at any order is finite in a suitable Sobolev norm.

We then consider in Sections 4 and 5 a modified Chapman–Enskog expansion of the kind considered by Caflisch [Ca87] and show also in this case that the remainder at any order is finite in the same Sobolev norm. The term of zero order in this expansion is a Maxwellian with parameters solving a set of dissipative new PDE's, the Vlasov–Navier–Stokes (VNS) equations, where, beyond the usual terms present in the compressible Navier–Stokes equations, there are diffusive terms coming from the presence of the self-consistent force. In particular, the equation for the concentration can be put in the form of a gradient flux of an energy functional [BELMII] which is similar to an exact evolution equation derived for a microscopic model of a binary alloy. The latter has been proven to yield the same late time phase segregation behavior as the Cahn–Hilliard equation, [GL96, GL97]. Both Vlasov–Euler and Vlasov–Navier–Stokes have non trivial stationary solutions with the same solitonic profile as in the BV equation.

Finally in Section 6 we consider the incompressible regime for these equations and derive, under suitable initial conditions and scaling, a set of

PDE's with dissipative terms involving a force linear in the concentration (they are essentially the linearization of the analogous terms in the compressible equations around a constant concentration and density profile). Above results all rely on the crucial assumption that the initial value problems for the hydrodynamical equations have a unique smooth solution at least on some macroscopic time interval. We do not discuss the technical conditions which ensure the existence of such solutions.

2. VLASOV–BOLTZMANN EQUATION FOR A BINARY MIXTURE

We consider a system of N_r *red* particles with positions ξ_i^r and velocities v_i^r, $i=1,..., N_r$ and N_b *blue* particles with positions ξ_i^b and velocities v_i^b, $i=1,..., N_b$, in a 3-dimensional torus Λ, interacting via two body forces. $N=N_r+N_b$ is the total number of particles. The potential energy is

$$V(\xi_1^r,..., \xi_{N_r}^r; \xi_1^b,..., \xi_{N_b}^b)$$
$$= \tfrac{1}{2} A_\ell \sum_{\alpha\neq\beta} \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} U_\ell(|\xi_i^\alpha - \xi_j^\beta|) + \tfrac{1}{2} \sum_{\alpha,\beta} \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} W_a(|\xi_i^\alpha - \xi_j^\beta|) \tag{2.1}$$

where $\alpha, \beta = r, b$, U_ℓ is the long range potential

$$U_\ell(r) = U\left(\frac{r}{\ell}\right) \tag{2.2}$$

for some bounded, smooth non-negative function U on $\mathbb{R}_+$. The factor A_ℓ is the intensity of the long range interaction to be suitably chosen to get a mean field type of behavior when ℓ becomes very large compared to the interparticle spacing [LP]. The potential W_a is the *formal* hard core potential

$$\mathscr{W}_a(r) = \begin{cases} \infty & \text{if} \quad r<a \\ 0 & \text{otherwise} \end{cases}$$

In other words the particles are hard spheres of diameter a interacting by elastic collisions which are color blind and by a weak repulsive long range force between particles of different species. The total number of particles of each species as well as the total momentum and energy are invariant during the evolution.

Choosing the size of Λ to be ℓ (or some constant multiple thereof) there are two characteristic length scales for this dynamics: a, the range of the hard core potential and ℓ, the range of the Kac potential. We can consider a third length, which depends on the density, the mean free path λ defined by the relation

$$\lambda = \frac{\ell^3}{Na^2}$$

The kinetic limit arises when there is a large separation between a and λ, corresponding to a low density (N/ℓ^3) situation. To obtain a kinetic limit we send N and ℓ to ∞ while a is fixed, say 1, in such a way that λ/ℓ is finite $(N \sim \ell^2)$ and assume initial data almost constant on regions of size $\sim \lambda$. We denote by

$$\delta = \frac{a}{\lambda} = \frac{1}{\lambda}$$

and assume finite

$$\gamma = \frac{\lambda}{\ell}$$

The kinetic equations will be obtained in the limit $\delta \to 0$, assuming

$$A_\ell = \gamma^3 \delta^2$$

meaning that A_ℓ is proportional to $1/N$. A further limit $\gamma \to 0$ will provide the hydrodynamical limit to be discussed later.

In kinetic coordinates q, that is $q_i^\alpha = \delta \xi_i^\alpha$, for $\alpha = r, b$ and $i = 1,..., N_\alpha$ and kinetic time $\tau = \delta \tau_m$, τ_m being the microscopic time, the equations of motion for the system are, for $\alpha = r, b$ and $i_\alpha = 1,..., N_\alpha$

$$\begin{aligned} \frac{dq_{i_\alpha}^\alpha}{d\tau} &= v_{i_\alpha}^\alpha \\ \frac{dv_{i_\alpha}^\alpha}{d\tau} &= \gamma^3 \delta^2 \sum_{j_\beta = 1}^{N_\beta} K(\gamma \, |q_{i_\alpha}^\alpha - q_{j_\beta}^\beta|)(1 - \delta_{\alpha\beta}) \end{aligned} \tag{2.3}$$

in

$$\Gamma_N = \{(q_1, v_1,..., q_N, v_n) \in \Lambda^N \times \mathbb{R}^{3N} \mid |q_i - q_j| > \delta, \; i \neq j\}$$

where $K(|x-y|) = -(\nabla U)(|x-y|)$, $N = N_r + N_b$, and we use the notation q_i, v_i, $i = 1,..., N$ when the color is irrelevant. When two particles are in contact (namely at distance δ) they undergo an elastic collision regardless of their color. We neglect the event that more that two-particles are in contact because it has vanishing Lebesgue measure. Hence the evolution (2.3) is defined only almost everywhere.

Note that when $N \sim \ell^2$ then the mean force on each particle on the kinetic scale (2.3) is of order unity. This is the reason why our original choice of the strength A_ℓ of the potential in (2.2) was like ℓ^{-2} rather than ℓ^{-3} as in the usual case [LP].

With this scaling we can get, at least formally, in the limit $\delta \to 0$ the Vlasov–Boltzmann equation for a binary mixture of hard core particles interacting via a weak long range potential. A formal proof of this is given in Appendix C. The rigorous proof would require the extension of the Lanford argument to this case, an extension that is not obvious because the Vlasov part is not well controlled in the Lanford norms.

Even if we have discussed the derivation of the Vlasov–Boltzmann equation only for hard spheres, from now on we consider the Vlasov–Boltzmann equations in full generality. The function $f^r(q, v, \tau)$ (resp. $f^b(q, v, \tau)$) is proportional to the probability density of finding a red (resp. blue) particle at $q \in \Omega \subset \mathbb{R}^3$, with velocity $v \in \mathbb{R}^3$ at time $\tau \geqslant 0$. We notice that the relation between the f^α's and the microscopic one particle densities $\rho_1^\alpha(\xi, v, \tau_m)$ (normalized to N_α) is given by

$$f^\alpha(q, v, \tau) = \lim_{\delta \to 0} \delta^{-1} \rho_1^\alpha(\delta^{-1} q, v, \delta^{-1} \tau)$$

The functions f^r and f^b are positive and normalized to γ^{-3} for any value of τ. They are solutions to the equations

$$\begin{aligned} \partial_\tau f^r + v \cdot \nabla_q f^r + F^r \cdot \nabla_v f^r &= J(f^r, f^r) + J(f^r, f^b) \\ \partial_\tau f^b + v \cdot \nabla_q f^b + F^b \cdot \nabla_v f^b &= J(f^b, f^b) + J(f^b, f^r) \end{aligned} \tag{2.4}$$

The Vlasov force acting on each particle is of the Kac type, meaning that for any $\gamma > 0$, the forces are conservative non local forces with range γ^{-1} defined by the position

$$F^\alpha(q, \tau) = -\nabla_q \int_\Omega dq' \, \gamma^3 U(\gamma \, |q - q'|) \, n^\beta(q', \tau), \qquad \alpha = r, b, \qquad \alpha \neq \beta \tag{2.5}$$

with $U(|q|)$ a smooth, non negative function of compact support and n^r, n^b are the rescaled spatial densities of the red and blue particles:

$$n^\alpha(q, \tau) = \int_{\mathbb{R}^3} dv \, f^{(\alpha)}(q, v, \tau), \qquad \int_\Omega dq \, n^{(\alpha)}(q, \tau) = \gamma^{-3} \tag{2.6}$$

For any positive functions f and g, $J(f, g)$ denotes the effect of the collisions of particles distributed according to g on the distribution f. Its expression is given by

$$J(f, g) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega \, b(|v - v_*|, \omega)[f(v') \, g(v'_*) - f(v) \, g(v_*)] \tag{2.7}$$

Here $b(|v|, \omega)$ is the differential cross section of the short range interaction, $\omega \in S_2$ is the impact parameter and v', v'_* are the incoming velocities corresponding to an elastic collision with outgoing velocities v, v_* and impact parameter ω. We assume the Grad's (see [Gra]) angular cutoff condition that $b(|v|, \omega)$ is a smooth function growing at most linearly for large $|v|$, i.e., $b(|v|, \omega) = |v|^\sigma h(\omega)$ with $0 \leqslant \sigma \leqslant 1$ and h a smooth bounded function on S_2.

An important property of the collisions is the entropy production inequality: let

$$\mathcal{N}_\alpha = \int_{\mathbb{R}^3} dv \, J(f^\alpha, f^\alpha) \log f^\alpha, \qquad \alpha = 1, 2$$

$$\mathcal{N}_{\alpha, \beta} = \int_{\mathbb{R}^3} dv \, J(f^\alpha, f^\beta) \log f^\alpha, \qquad \alpha, \beta = 1, 2$$

Then $\mathcal{N}_\alpha$'s as well as $\mathcal{N}_{1,2} + \mathcal{N}_{2,1}$ are non negative. Moreover N_α vanishes as usual if and only if the f^α's are Maxwellians:

$$f^\alpha = M(n^\alpha, u^\alpha, T^\alpha; v), \qquad \alpha = 1, 2$$

with

$$M(n, u, T; v) := \frac{n}{(2\pi T)^{3/2}} \, \mathrm{e}^{-(v-u)^2/2T} \tag{2.8}$$

Furthermore $\mathcal{N}_{1,2} + \mathcal{N}_{2,1}$ vanishes if and only if the two Maxwellians have the same local temperature and mean velocities:

$$u^\alpha = u, \qquad T^\alpha = T, \qquad \alpha = 1, 2$$

This implies that the only solutions of the equations

$$J(f_1, f_1) + J(f_1, f_2) = 0$$

$$J(f_2, f_2) + J(f_2, f_1) = 0$$

are Maxwellians with the same mean velocity and temperature. General arguments suggest that all the stationary solutions of Eqs. (2.4) will be Maxwellians with $u=0$, $T(q)=T$, and densities satisfying the equations

$$\begin{aligned} T\log n^1(q)+\int dq'\, \gamma^3 U(\gamma\,|q-q'|)\, n^2(q') &= C_1 \\ T\log n^2(q)+\int dq'\, \gamma^3 U(\gamma\,|q-q'|)\, n^1(q') &= C_2 \end{aligned} \tag{2.9}$$

Beyond the spatially constant equilibria, there may be other spatially non homogeneous solutions. For example, by prescribing the boundary conditions in one dimension

$$\lim_{z\to\pm\infty} n^\alpha(z)=\bar{n}^\alpha_\pm$$

one gets at small values of T a solitonic solution describing the interface profile [BL]. We shall leave a discussion of this part for [BELMII] and focus here on deriving macroscopic equations for the evolution of the conserved quantities.

Before closing this section, let us define

$$f(q,v,\tau)=\tfrac{1}{2}[f^r(q,v,\tau)+f^b(q,v,\tau)]$$

as the density of finding a particle at q with velocity v at time τ, independently of its color. Moreover, we set

$$\phi(q,v,\tau)=\tfrac{1}{2}[f^r(q,v,\tau)-f^b(q,v,\tau)]$$

The system (2.4) can be written in the following equivalent form:

$$\begin{aligned} \partial_\tau f+v\cdot\nabla_q f+2F\cdot\nabla_v f+2W\cdot\nabla_v\phi &= 4J(f,f) \\ \partial_\tau \phi+v\cdot\nabla_q \phi+2F\cdot\nabla_v \phi+2W\cdot\nabla_v f &= 4J(\phi,f) \end{aligned} \tag{2.10}$$

where $F=F^r+F^b$, $W=F^r-F^b$. We can absorb the numerical factors by redefining U as $U/2$ and b as $b/4$ so obtaining

$$\begin{aligned} \partial_\tau f+v\cdot\nabla_q f+F\cdot\nabla_v f+W\cdot\nabla_v\phi &= J(f,f) \\ \partial_\tau \phi+v\cdot\nabla_q \phi+F\cdot\nabla_v \phi+W\cdot\nabla_v f &= J(\phi,f) \end{aligned} \tag{2.11}$$

3. COMPRESSIBLE HYDRODYNAMICS

We are interested in the behavior of the system on the macroscopic scale. To this end we introduce a scaling parameter ε representing the ratio between the kinetic and macroscopic space units and, for any $t \geqslant 0$ and $x \in \varepsilon\Omega$ we set

$$\tau = \varepsilon^{-1} t, \qquad q = \varepsilon^{-1} x$$

We assume that at time zero the densities vary slowly on the microscopic scale $f^i(q, v, 0) = \tilde{f}(\varepsilon q, v, 0)$ and look for solutions of (2.11) such that

$$f^i(q, v, \tau) = \tilde{f}^i(\varepsilon q, v, \varepsilon\tau), \qquad i = r, b$$

with $\tilde{f}^i$ smooth functions on $\varepsilon\Omega \times \mathbb{R}^3 \times \mathbb{R}_+$. For the force we have

$$F^r(\varepsilon^{-1}x, \varepsilon^{-1}t) = -\varepsilon \nabla_x \int_{\varepsilon\Omega} dx' \left(\frac{\gamma}{\varepsilon}\right)^d U\left[\left(\frac{\gamma}{\varepsilon}\right)|x - x'|\right] \tilde{n}^b(x', t)$$

and a similar relation for F^b. Therefore, if we assume $\gamma = \varepsilon$, also the forces are slowly varying functions and the $\tilde{f}$ satisfy the following system, where we remove the "tilde's" because in the sequel we shall always consider only the macroscopic variables:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f + W \cdot \nabla_v \phi = \varepsilon^{-1} J(f, f) \\ &\partial_t \phi + v \cdot \nabla_x \phi + F \cdot \nabla_v \phi + W \cdot \nabla_v f = \varepsilon^{-1} J(\phi, f) \end{aligned} \tag{3.1}$$

We shall use the notation

$$F = \mathbf{K} \circledast f, \qquad W = -\mathbf{K} \circledast \phi \tag{3.2}$$

where,

$$\mathbf{K}(x) = -\nabla_x U(|x|) \tag{3.3}$$

and, for any function g we set

$$(\mathbf{K} \circledast g)(x, t) \stackrel{\text{def}}{=} \int_{\Omega} dx' \, \mathbf{K}(|x - x'|) \int_{\mathbb{R}^3} dv \, g(x', v, t) \tag{3.4}$$

We will show that the solution of the system (3.1) is close for ε small to the local equilibrium with parameters $\rho^{(1)}$, $\rho^{(2)}$, u and T satisfying the following set of hydrodynamic equations:

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot [\rho u] = 0\\
&\partial_t \varphi + \nabla \cdot (\varphi u) = \varepsilon \nabla \cdot (DQ)\\
&\rho D_t u + \nabla P - \rho \mathbf{K} * \rho + \varphi \mathbf{K} * \varphi = -\varepsilon \nabla \sigma\\
&\tfrac{3}{2} \rho D_t T + P \nabla \cdot u = \varepsilon \nabla(\kappa \nabla T) - \varepsilon \sigma : \nabla u - \varepsilon \mathbf{K} * \varphi \cdot DQ
\end{aligned}
\tag{3.5}
$$

Here $\rho = \rho^{(1)} + \rho^{(2)}$ is the total density, $\varphi = \rho^{(1)} - \rho^{(2)}$, $P = \rho T$

$$
\begin{aligned}
&D_t := \partial_t + u \cdot \nabla\\
&\sigma := -\nu \left(\nabla u + \nabla u^\dagger - \frac{2}{3} \mathbb{I} \, \nabla \cdot u \right)\\
&Q := \nabla \frac{\varphi}{\rho} + \frac{1}{\rho^2 T} (\rho^2 - \varphi^2) \, \mathbf{K} * \varphi
\end{aligned}
\tag{3.6}
$$

$\nabla u^\dagger$ is the adjoint of the matrix ∇u, $\sigma : \nabla u = \mathrm{Tr}(\sigma \nabla u)$, $\mathbb{I}$ is the unit matrix, ν and $D\rho$ are the viscosity and the diffusion coefficients and κ is the heat conductivity. These are computed from the VBE. The above equations, with $\varepsilon = 0$ will be referred to as the Vlasov–Euler equations (VE). We assume that the initial value problem for such equations, with suitable initial data, has a sufficiently smooth solution at least on a time interval $[0, \bar{t}]$. Under such conditions we will prove in the next section and in Appendix A that the solution to the VBE for the binary fluid, under the Euler scaling, converges to the Maxwellian local equilibrium with parameters satisfying the VE equations in the interval $[0, \bar{t}]$, with an error of order ε (Proposition 4.1 and Corollary 4.2).

When $\varepsilon > 0$, the above equations will be referred to as the Vlasov–Navier–Stokes equations (VNS). In Section 5, using also the arguments of Appendix A we will show that their solutions provide an approximation up to the order ε^2 to the solutions of the VBE in the Euler scaling, provided that the initial value problem for such equations has suitably smooth solutions as before. The precise statement is given in Proposition 5.1 and Corollary 5.2.

Like for the usual Navier–Stokes equations, which are frequently and successfully used with $\varepsilon = 1$ in physical and engineering applications, although their derivation is restricted to small values of ε, we will consider the VNS equations with $\varepsilon = 1$ and analyze some of their properties in [BELMII]. In order to get diffusive effects as sharp limits of the VBE, it is necessary to go to the parabolic scaling where $\tau = \varepsilon^{-2} t$ and consider simultaneously a low Mach number situation. This will be discussed in Section 6.

4. EULER LIMIT

We outline the proof of the convergence of the Vlasov–Boltzmann system to the VE equations. The proof will be completed in Appendix A. We fix the Maxwellian $M(\rho, u, t)$ with ρ, u, T possibly depending on space and time and denote

$$\mathscr{L}f = J(M, f) + J(f, M) \tag{4.1}$$

and

$$\Gamma f = J(f, M) \tag{4.2}$$

Moreover, we set

$$Q(f, g) = \tfrac{1}{2}[J(g, f) + J(f, g)] \tag{4.3}$$

It is easy to check that, as for the one-component Boltzmann equation,

$$\mathscr{L}f = 0 \qquad \text{iff} \quad f = M\chi_\alpha, \qquad \alpha = 0,..., 4 \tag{4.4}$$

where

$$\chi_0 = 1, \qquad \chi_i = v_i, \qquad i = 1,..., 3, \qquad \chi_4 = v^2/2 \tag{4.5}$$

Moreover, along the same lines one gets

$$\Gamma f = 0 \qquad \text{iff} \quad f = aM, \qquad a \in \mathbb{R} \tag{4.6}$$

We shall try to solve (3.1) following [Ca80], in terms of a truncated Hilbert expansion of the form

$$\begin{aligned} f &= \sum_{n=0}^{K} \varepsilon^n f_n + \varepsilon^m R_f \\ \phi &= \sum_{n=0}^{K} \varepsilon^n \phi_n + \varepsilon^m R_\phi \end{aligned} \tag{4.7}$$

with suitably chosen positive integers K and m. The functions f_n and ϕ_n are computed using a Hilbert expansion and the remainders R_f and R_ϕ are defined as the difference between the solution and the truncated expansion.

We substitute in (3.1) the formal power series

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_n, \qquad \phi = \sum_{n=0}^{\infty} \varepsilon^n \phi_n \tag{4.8}$$

$$F = \sum_{n=0}^{\infty} \varepsilon^n F_n = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{K} \circledast f_n, \qquad W = \sum_{n=0}^{\infty} \varepsilon^n W_n = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{K} \circledast \phi_n \tag{4.9}$$

and denote by D_t the time derivative along the trajectories:

$$D_t = \partial_t + v \cdot \nabla_x$$

We have:

$$\varepsilon^{-1} Q(f_0, f_0) + \sum_{n=0}^{\infty} \varepsilon^n [2Q(f_0, f_{n+1}) + \mathscr{S}_n] = 0 \tag{4.10}$$

$$\varepsilon^{-1} J(\phi_0, f_0) + \sum_{n=0}^{\infty} \varepsilon^n [J(\phi_{n+1}, f_0) + J(\phi_0, f_{n+1}) + \mathscr{T}_n] = 0 \tag{4.11}$$

where

$$\mathscr{S}_n = \sum_{\substack{(h,h'):\, h,h' \geqslant 1 \\ h+h'=n+1}} Q(f_h, f_{h'}) - \sum_{\substack{(h,h'):\, h,h' \geqslant 0 \\ h+h'=n}} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}] - D_t f_n \tag{4.12}$$

$$\mathscr{T}_n = \sum_{\substack{(h,h'):\, h,h' \geqslant 1 \\ h+h'=n+1}} J(\phi_h, f_{h'}) - \sum_{\substack{(h,h'):\, h,h' \geqslant 0 \\ h+h'=n}} [F_h \cdot \nabla_v \phi_{h'} + W_h \cdot \nabla_v f_{h'}] - D_t \phi_n \tag{4.13}$$

In order that the formal series solve (3.1) the coefficients have to satisfy the conditions:

$$\begin{aligned} Q(f_0, f_0) &= 0 \\ J(\phi_0, f_0) &= 0 \end{aligned} \tag{4.14}$$

and, for any $n \geqslant 0$,

$$\begin{aligned} 2Q(f_0, f_{n+1}) + \mathscr{S}_n &= 0 \\ J(\phi_{n+1}, f_0) + J(\phi_0, f_{n+1}) + \mathscr{T}_n &= 0 \end{aligned} \tag{4.15}$$

As remarked in the previous section, the first of the conditions (4.14), implies that f_0 is a Maxwellian with parameters depending on x, t:

$$f_0(x, v, t) = M(\rho(x,t), u(x,t), T(x,t); v) := M(v) \tag{4.16}$$

Moreover from the second equation of (4.14) we get

$$\phi_0(x, v, t) = \frac{\varphi(x, t)}{\rho(x, t)} M(\rho(x, t), u(x, t), T(x, t); v)$$

for some suitable function $\varphi(x, t)$. Using (4.1) and (4.2) we can write (4.15) as

$$\begin{aligned} \mathscr{L} f_{n+1} &= -\mathscr{S}_n \\ \Gamma \phi_{n+1} &= -J(\phi_0, f_{n+1}) - \mathscr{T}_n \end{aligned} \tag{4.17}$$

Since $\mathscr{S}_n$ and $\mathscr{T}_n$ only depend on the f_k and ϕ_k for $k \leqslant n$, we have first to solve the first equation and then, once f_{n+1} is determined, we solve the second one for ϕ_{n+1}.

In order to check the solvability of these equations we introduce the Hilbert space $\mathscr{H}$ of measurable functions on $\mathbb{R}^3$ such that the scalar product

$$(f, g) = \int_{\mathbb{R}^3} dv\, f(v)\, g(v)\, M^{-1}(v) \tag{4.18}$$

is finite. In this Hilbert space the operators $\mathscr{L}$ and Γ are densely defined and symmetric. Moreover, the null spaces of $\mathscr{L}$ and Γ are the five-dimensional subspace spanned by $\{M\chi_\alpha, \alpha = 0,..., 4\}$ introduced in (4.5) and the one-dimensional space spanned by $M\chi_0$ respectively. We denote by $\mathscr{P}$ and $\mathscr{K}$ the projectors on such subspaces and by $\mathscr{P}^\perp = 1 - \mathscr{P}$ and $\mathscr{K}^\perp = 1 - \mathscr{K}$ the projectors on their orthogonal complements. From the properties of $\mathscr{L}$ and Γ it is immediate to check that

$$\mathscr{P}\mathscr{L} = 0, \qquad \mathscr{K}\Gamma = 0 \tag{4.19}$$

Both $\mathscr{L}$ and Γ are non positive and there are positive constants δ and δ'

$$(f, \mathscr{L}f) \leqslant -\delta\, \|\mathscr{P}^\perp f\|^2, \qquad (f, \Gamma f) \leqslant -\delta'\, \|\mathscr{K}^\perp f\|^2 \tag{4.20}$$

Moreover

$$\begin{aligned} \mathscr{L} &= -\nu + K \\ \Gamma &= -\nu + \Theta \end{aligned} \tag{4.21}$$

where

$$\nu(x, v, t) = \int_{Re^3} dv_* b(|v - v_*|) M(\rho(x, t), u(x, t), T(x, t); v_*) \tag{4.22}$$

is a strictly positive function such that

$$\nu_0 (1 + |v|)^\sigma \leqslant \nu(x, v, t) \leqslant \nu_1 (1 + |v|)^\sigma \tag{4.23}$$

for some positive constants ν_0 and ν_1 provided that ρ and T are bigger than some fixed positive constants. Furthermore, K and Θ are compact operators on $\mathscr{H}$. Therefore, using the Fredholm alternative theorem we can conclude the existence of solutions to (4.17) provided that $\mathscr{S}_n \in \mathscr{P}^\perp \mathscr{H}$ and $\mathscr{T}_n + J(\phi_0, f_{n+1}) \in \mathscr{K}^\perp \mathscr{H}$. These conditions can be verified inductively, as in the usual one-component Boltzmann equation. We now write the conditions for $n = 0$ which determine the macroscopic equations for ρ, u, T and φ: Since $\mathscr{P}[\nabla_v f_0]$ and $\mathscr{P}[\nabla_v \phi_0]$ have no component along χ_0, it is easy to check that the condition $\mathscr{P}\mathscr{S}_0 = 0$ can be written explicitly as

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot [\rho u] &= 0 \\ \rho[\partial_t u + (u \cdot \nabla_x) u] &= -\nabla_x P + \rho \mathbf{K} * \rho - \varphi \mathbf{K} * \varphi \\ \rho[\partial_t e + (u \cdot \nabla_x) e] + P \nabla_x \cdot u &= 0 \end{aligned} \tag{4.24}$$

where $*$ denotes the usual convolution, $P = \rho T$ is the equation of state for the pressure in the perfect gas and $e = 3T/2$ is its internal kinetic energy. On the other hand, since $\mathscr{K} J = 0$ and $\mathscr{K} \nabla_v = 0$, the condition $\mathscr{K}\mathscr{T}_0 = 0$ becomes $\mathscr{K} D_t \phi_0 = 0$ which is explicitly written as

$$\partial_t \varphi + \nabla_x \cdot [\varphi u] = 0 \tag{4.25}$$

Equations (4.24) and (4.25) represent the Euler equations for the binary mixture. They differ from the usual Euler equations by the presence of the equation (4.25) for φ and for the nonlinear self consistent force terms due to the long range Kac interaction. We will refer to them as the *Vlasov–Euler equations* (*VE*). Existence of solutions to the initial value problem for the system (4.24)–(4.25) requires some analysis but we do not discuss this. We simply assume that, for sufficiently smooth initial data a unique solution of the system exists and stays smooth up to some time $\bar{t}$.

Given such a solution, the functions f_1 and ϕ_1 can be found by solving (4.17) with $n = 0$. In consequence, f_1 is determined up to $p_1 \in \mathscr{P}\mathscr{H}$ and ϕ_1 up to $q_1 \in \mathscr{K}\mathscr{H}$. The procedure can then continue by taking advantage of the arbitrariness of p_1 and q_1 to satisfy the conditions $\mathscr{P}\mathscr{S}_1 = 0$, $\mathscr{K}\mathscr{T}_1 = 0$.

In this way the functions f_n and ϕ_n can be found for any n. Classical results by Grad [Gra] provide the smoothness and decay properties we use below.

Now we go back to the truncated expansions (4.7). Once φ_n and ϕ_n are computed for $n=0,\ldots, K$, we can look for the equations for the remainders R_f and R_ϕ. A straightforward calculation shows that, in order that f and ϕ satisfy (3.1), R_f and R_ϕ have to solve the equations

$$\begin{aligned} &D_t R_f + F\cdot\nabla_v R_f + W\cdot\nabla_v R_\phi \\ &\qquad = \varepsilon^{-1}\mathscr{L}R_f + \mathscr{L}^{(1)}R_f + \varepsilon^{m-1}[J(R_f, R_f) + A_f] \\ &D_t R_\phi + F\cdot\nabla_v R_\phi + W\cdot\nabla_v R_f \\ &\qquad = \varepsilon^{-1}\Gamma R_\phi + \varepsilon^{-1}\tilde{\Theta}R_f + \Gamma^{(1)}R\phi + \varepsilon^{m-1}[J(R_\phi, R_f) + A_\phi] \end{aligned} \tag{4.26}$$

where

$$\begin{gathered} \mathscr{L}^{(1)}g = \sum_{h=1}^{K}\varepsilon^{h-1}[J(f_h, g) + J(g, f_h)] \\ \tilde{\Theta}g = J\left(\sum_{n=0}^{K}\varepsilon^n\phi_n, g\right), \qquad \Gamma^{(1)}g = J\left(g, \sum_{h=1}^{K}\varepsilon^{h-1}f_h\right) \end{gathered} \tag{4.27}$$

$$\begin{aligned} A_f = \varepsilon^{K-2m+1}\Bigg(&\sum_{\substack{(h,h'):\,h,h'\geqslant 1\\ h+h'>K+1}}\varepsilon^{h+h'-K-1}Q(f_h, f_{h'}) \\ &-\sum_{\substack{(h,h'):\,h,h'\geqslant 0\\ h+h'>K}}\varepsilon^{h+h'-K}[F_h\cdot\nabla_v f_{h'} + W_h\cdot\nabla_v\phi_{h'}] - D_t f_K\Bigg) \\ A_\phi = \varepsilon^{K-2m+1}\Bigg(&\sum_{\substack{(h,h'):\,h,h'\geqslant 1\\ h+h'>K+1}}\varepsilon^{h+h'-K-1}J(\phi_h, f_{h'}) \\ &-\sum_{\substack{(h,h'):\,h,h'\geqslant 0\\ h+h'>K}}\varepsilon^{h+h'-K}[W_h\cdot\nabla_v f_{h'} + F_h\cdot\nabla_v\phi_{h'}] - D_t\phi_K\Bigg) \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} F &= \sum_{n=0}^{K}\varepsilon^n F_n + \varepsilon^m\mathbf{K}\circledast R_f \\ W &= \sum_{n=0}^{K}\varepsilon^n W_n + \varepsilon^m\mathbf{K}\circledast R_\phi \end{aligned} \tag{4.29}$$

The expressions of A_f and A_ϕ show that it is convenient to choose $K \geqslant 2m-1$ in order to get them bounded as $\varepsilon \to 0$.

The construction of the solution R_f, R_ϕ of (4.26) is obtained using a fixed point argument to handle the nonlinear terms in the equations.

In Appendix A we shall sketch the proof of the Proposition below, which extends the similar result proved for the one-component Boltzmann gas without long-range interactions in [Ca80] and [Lac]. We assume for simplicity periodic boundary conditions, namely Ω is the 3-dimensional torus of unit side. The potential of the long range force is assumed C^∞, non negative and of compact support. Of course such assumptions could be relaxed, but we will not try to examine the most general setup. Moreover we use the norm

$$\|f\|_{\alpha, \ell, s} = \sup_{v \in \mathbb{R}^3} [e^{\alpha v^2}(1+|v|^2)^{\ell/2} \, |f(\cdot, v)|_s] \tag{4.30}$$

and $|f|_s$ is the Sobolev norm of order s.

We will refer below to sufficiently smooth solutions to the VE equations meaning solutions which are in H_s for some sufficiently large s and such that the inequalities

$$T_0 \leqslant T \leqslant T_1, \qquad \rho_0 \leqslant \rho \pm \varphi \leqslant \rho_1$$

are verified for suitable positive constants T_0, T_1, ρ_0 and ρ_1.

Proposition 4.1. Suppose that (ρ, u, T, φ) is a solution to the Vlasov–Euler Eqs. (4.24), (4.25) sufficiently smooth in the time interval $[0, \bar{t}\,]$. Then there are positive constants ε_0 and C such that, for $\varepsilon < \varepsilon_0$ a unique classical solution to the system (4.26) with $m \geqslant 4$ exists and satisfies the bounds

$$\begin{aligned} \sup_{t \in [0, \bar{t}\,]} \|R_f(\cdot, t)\|_{\alpha, \ell, s} &\leqslant C\varepsilon \sup_{t \in [0, \bar{t}\,]} [\|A_f(\cdot, t)\|_{\alpha, \ell, s} + \|A_\phi(\cdot, t)\|_{\alpha, \ell, s}] \\ \sup_{t \in [0, \bar{t}\,]} \|R_\phi(\cdot, t)\|_{\alpha, \ell, s} &\leqslant C\varepsilon \sup_{t \in [0, \bar{t}\,]} [\|A_f(\cdot, t)\|_{\alpha, \ell, s} + \|A_\phi(\cdot, t)\|_{\alpha, \ell, s}] \end{aligned} \tag{4.31}$$

for any positive $\alpha < \bar{T}/2$, $\bar{T} \stackrel{\text{def}}{=} \sup_{x \in \Omega, t \in [0, \bar{t}\,]} T(x, t)$, $\ell > 3$, $s \geqslant 2$.

Corollary 4.2. Under the assumptions of Proposition 3.1, for $\varepsilon < \varepsilon_0$ there is a smooth solution $(f^\varepsilon_t, \phi^\varepsilon_t)$ to the rescaled Vlasov–Boltzmann equations (3.1) and moreover, denoting by M_t the Maxwellian with parameters evolving according to the Euler equations, it satisfies:

$$\sup_{0\leqslant t\leqslant \bar{t}} \left[\|f_t^\varepsilon - M_t\|_{\alpha,\ell,s} + \left\| \phi_t^\varepsilon - \frac{\varphi_t}{\rho_t} M_t \right\|_{\alpha,\ell,s} \right] \leqslant C\varepsilon$$

5. NAVIER STOKES CORRECTION

The Navier–Stokes corrections to the hydrodynamical equations on the Euler scale are usually obtained by means of a suitable resummation of the Hilbert series expansion called the *Chapman–Enskog expansion*. For our purposes it is convenient to look at a modified version of the expansion proposed by Caflisch [Ca87].

We use the notation: for $n\geqslant 0$, $\hat{f}_n = \mathscr{P} f_n$, $\bar{f}_n = \mathscr{P}^\perp f_n$, $\hat{\phi}_n = \mathscr{K}\phi_n$, $\bar{\phi}_n = \mathscr{K}^\perp \phi_n$, $F_n = \mathbf{K} \circledast f_n$, $W_n = \mathbf{K} \circledast \phi_n$. The terms in the expansions are given as follows: we set $M_s := M(1, u, T)$;

$$f_0 = \rho M_s, \qquad \phi_0 = \varphi M_s, \qquad \hat{f}_1 = 0, \qquad \hat{\phi}_1 = 0 \tag{5.1}$$

$$\mathscr{L}\bar{f}_1 = \mathscr{P}^\perp [D_t f_0 + F_0 \cdot \nabla_v f_0 + W_0 \cdot \nabla_v \phi_0] \tag{5.2}$$

$$\Gamma\bar{\phi}_1 = -J(\phi_0, \bar{f}_1) + \mathscr{K}^\perp \left[D_t \phi_0 - \varepsilon \frac{\varphi}{\rho^2} \mathscr{P}[D_t \bar{f}_1] + W_0 \cdot \nabla_v f_0 + F_0 \cdot \nabla_v \phi_0 \right] \tag{5.3}$$

$$\mathscr{P}[D_t(f_0 + \varepsilon\bar{f}_1) + F_0 \cdot \nabla_v (f_0 + \varepsilon\bar{f}_1) + W_0 \cdot \nabla_v (\phi_0 + \varepsilon\bar{\phi}_1)] = 0 \tag{5.4}$$

$$\mathscr{K}[D_t(\phi_0 + \varepsilon\bar{\phi}_1) + W_0 \cdot \nabla_v (f_0 + \varepsilon\bar{f}_1) + F_0 \cdot \nabla_v (\phi_0 + \varepsilon\bar{\phi}_1)] = 0 \tag{5.5}$$

$$\begin{aligned} \mathscr{L}\bar{f}_2 = {} & -2Q(f_1, f_1) + \mathscr{P}^\perp [D_t(\bar{f}_1 + \varepsilon\hat{f}_2) \\ & + [F_0 \cdot \nabla_v f_1 + F_1 \cdot \nabla_v f_0 + W_0 \cdot \nabla_v \phi_1 + W_1 \cdot \nabla_v \phi_0]] \end{aligned} \tag{5.6}$$

$$\begin{aligned} \Gamma\bar{\phi}_2 = {} & -\varphi J(M_s, f_2) - J(\phi_1, f_1) + \mathscr{K}^\perp \Bigg[D_t \bar{\phi}_n + \frac{\varphi}{\rho^2} \mathscr{P}[D_t \bar{f}_1] + \varepsilon D_t \hat{\phi}_2 \\ & + [W_0 \cdot \nabla_v f_1 + W_1 \cdot \nabla_v f_0 + F_0 \cdot \nabla_v \phi_1 + F_1 \cdot \nabla_v \phi_0] \Bigg] \end{aligned} \tag{5.7}$$

$$\mathscr{P}\left[D_t f_2 + \sum_{\substack{(h,h'):\, h\geqslant 0,\, h'>0 \\ h+h'=2}} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}] \right] = 0 \tag{5.8}$$

$$\mathscr{K}\left[D_t \phi_2 + \sum_{\substack{(h,h'):\, h\geqslant 0,\, h'>0 \\ h+h'=2}} [F_h \cdot \nabla_v \phi_{h'} + W_h \cdot \nabla_v f_{h'}] \right] = 0 \tag{5.9}$$

Before giving conditions defining the higher order terms of the expansion let us comment about previous conditions: Eqs. (5.1)–(5.9) have to be solved in the order they are written: (5.2) and (5.3) are used to determine $\bar{f}_1$ and then $\bar{\phi}_1$ in terms of the hydrodynamical parameters (ρ, u, T, φ) and their derivatives; as a consequence (5.4) and (5.5) only involve (ρ, u, T, φ) and represent the hydrodynamical equations we are looking for. Because of (5.1) f_1 and ϕ_1 are completely determined. Then (5.6) and (5.7) can be solved to find $\bar{f}_2$ and $\bar{\phi}_2$, depending only on f_0, f_1, ϕ_0, ϕ_1 and on the hydrodynamical parts of f_2 and ϕ_2. Finally, (5.8) and (5.9) are linear equations in the hydrodynamical part of f_2 and ϕ_2 which can be used to determine them.

We notice that the term proportional to ε in (5.3) has been included to avoid third order derivatives in the hydrodynamical equations. This is usually done in the standard Chapman–Enskog expansion by *expanding the time derivatives in powers of* ε.

For $n \geqslant 2$ we set:

$$\mathscr{L}\bar{f}_{n+1} = -\sum_{\substack{k, j \geqslant 1 \\ k+j=n+1}} 2Q(f_j, f_k) + \mathscr{P}^{\perp}\Bigg[D_t(\bar{f}_n + \varepsilon \hat{f}_{n+1}) + \sum_{\substack{(h, h') : h, h' \geqslant 0 \\ h+h'=n}} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}]\Bigg] \tag{5.10}$$

$$\Gamma\bar{\phi}_{n+1} = -\varphi J(M_s, f_{n+1}) - \sum_{\substack{k, j \geqslant 1 \\ k+j=n+1}} J(\phi_k, f_j) + \mathscr{K}^{\perp}\Bigg[D_t\bar{\phi}_n + \varepsilon D_t\hat{\phi}_{n+1} + \sum_{\substack{(h, h') : h, h' \geqslant 0 \\ h+h'=n}} [F_h \cdot \nabla_v \phi_{h'} + W_h \cdot \nabla_v f_{h'}]\Bigg] \tag{5.11}$$

$$\mathscr{P}\Bigg[D_t f_{n+1} + \sum_{\substack{(h, h') : h \geqslant 0, h' > 0 \\ h+h'=n+1}} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}]\Bigg] = 0 \tag{5.12}$$

$$\mathscr{K}\Bigg[D_t \phi_{n+1} + \sum_{\substack{(h, h') : h \geqslant 0, h' > 0 \\ h+h'=n+1}} [F_h \cdot \nabla_v \phi_{h'} + W_h \cdot \nabla_v f_{h'}]\Bigg] = 0 \tag{5.13}$$

The procedure used for f_2 and ϕ_2 can be repeated to get f_n and ϕ_n for any $n > 2$.

As for the Euler limit discussed in the previous section, instead of looking for the convergence of the expansion we consider its truncation

(4.7) where the functions f_n and ϕ_n are computed according to the procedure just explained, but setting $f_n = 0$, $\phi_n = 0$ for $n \geqslant K+1$. The remainders R_f and R_ϕ have to be solutions of the equations

$$\begin{aligned} &D_t R_f + F \cdot \nabla_v R_f + W \cdot \nabla_v R_\phi \\ &\qquad = \varepsilon^{-1} \mathscr{L} R_f + \mathscr{L}^{(1)} R_f + \varepsilon^{m-1} [J(R_f, R_f) + A_f] \\ &D_t R_\phi + W \cdot \nabla_v R_f + F \cdot \nabla_v R_\phi \\ &\qquad = \varepsilon^{-1} [\Gamma R_\phi + \tilde{\Theta} R_f] + \Gamma^{(1)} R_\phi + \varepsilon^{m-1} [J(R_\phi, R_f) + A_\phi)] \end{aligned} \tag{5.14}$$

where F and W are given by (4.29), $\mathscr{L}^{(1)}$, $\Gamma^{(1)}$ and $\tilde{\Theta}$ are given by (4.27), while the expressions of A_f and A_ϕ are slightly different from (4.28), and are:

$$\begin{aligned} A_f = \varepsilon^{K-2m+1} \Bigg(&\sum_{\substack{(h, h') : h, h' \geqslant 1 \\ h+h' > K+1}} \varepsilon^{h+h'-K-1} Q(f_h, f_{h'}) \\ &- \sum_{\substack{(h, h') : h, h' \geqslant 0 \\ h+h' > K}} \varepsilon^{h+h'-K} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}] \\ &- D_t \bar{f}_K - \mathscr{P}[F_0 \cdot \nabla_v f_K + W_0 \cdot \nabla_v \phi_K] \Bigg) \end{aligned} \tag{5.15}$$

$$\begin{aligned} A_\phi = \varepsilon^{K-2m+1} \Bigg(&\sum_{\substack{(h, h') : h, h' \geqslant 1 \\ h+h' > K+1}} \varepsilon^{h+h'-K-1} J(\phi_h, f_{h'}) \\ &- \sum_{\substack{(h, h') : h, h' \geqslant 0 \\ h+h' > K}} \varepsilon^{h+h'-K} [W_h \cdot \nabla_v f_{h'} + F_h \cdot \nabla_v \phi_{h'}] \\ &- D_t \bar{\phi}_K - \mathscr{K}[W_0 \cdot \nabla_v f_K + F_0 \cdot \nabla_v \phi_K] \Bigg) \end{aligned} \tag{5.16}$$

We now exploit the Eqs. (5.2)–(5.5) in order to obtain the Navier–Stokes equations for the binary mixture with long range forces.

Note that in (5.2) the force terms do not contribute because $\mathscr{P}^\perp \nabla_v M_s = 0$. Therefore $\bar{f}_1$ has the same expression as for the one component gas without self-interaction: recall that

$$\mathscr{P}^\perp (D_t f_0) = M \left[\sum_{i, j=1}^{3} A_{i, j} \partial_i u_j + \sum_{i=1}^{3} B_i \partial_i T \right] \tag{5.17}$$

with

$$A_{i,j} = \frac{1}{T}\left(\tilde{v}_i \tilde{v}_j - \frac{\tilde{v}^2}{3}\,\delta_{i,j}\right), \qquad B_i = \left(\frac{\tilde{v}^2}{2} - \frac{5}{2}\,T\right)\frac{\tilde{v}_i}{T^2}$$

and $\tilde{v} = v - u$.

Therefore

$$\bar{f}_1 = -\psi_1 \sum_{i,j=1}^{3} A_{i,j}\,\partial_i u_j - \psi_2 \sum_{i=1}^{3} B_i\,\partial_i T \tag{5.18}$$

with ψ_1 and ψ_2 non negative smooth functions of $|\tilde{v}|$. Moreover, $\mathscr{P}[\nabla_v \bar{f}_1] = 0$ because $\bar{f}_1$ is orthogonal to the invariants. On the other hand $(M_s\chi_\alpha, \nabla_v \bar{\phi}_1) = 0$ for $\alpha = 0,..., 3$, because $\bar{\phi}_1$ is orthogonal to the constants. Furthermore

$$\int dv [F_0 \cdot \nabla_v(f_0) + W_0 \cdot \nabla_v \phi_0] = 0 \tag{5.19}$$

$$\int dv\; \tilde{v}[F_0 \cdot \nabla_v(f_0) + W_0 \cdot \nabla_v \phi_0 = -F_0 \rho - W_0 \varphi \tag{5.20}$$

Hence the mass equation is

$$\partial_t \rho + \nabla \cdot (\rho u) = 0 \tag{5.21}$$

and the momentum equation is

$$\rho D_t^u u + \nabla P = -\varepsilon\, \nabla \cdot \sigma + \rho F_0 + \varphi W_0 \tag{5.22}$$

with $D_t^u = \partial_t + u \cdot \nabla$ and

$$\sigma_{i,j} := -\nu(\partial_j u_i + \partial_i u_j - \tfrac{2}{3}\delta_{i,j}\,\nabla \cdot u)$$

and

$$\nu = \int dv\; \psi_1(|\tilde{v}|)\; A_{1,2}^2$$

In order to compute the equation for the energy and for the concentration we need the expression of $\bar{\phi}_1$, which has to be computed using (5.3). This implies

$$\bar{\phi}_1 = \Gamma^{-1}\mathscr{K}^{\perp}\left[D_t\phi_0 - \varepsilon\frac{\varphi}{\rho^2}\mathscr{P}[D_t\bar{f}_1] + W_0\cdot\nabla_v f_0 + F_0\cdot\nabla_v\phi_0 - J(\phi_0, \bar{f}_1)\right] \tag{5.23}$$

We are interested in computing the component of $\nabla_v\bar{\phi}_1$ along $M\chi_4$ i.e., after an integration by parts:

$$(M\chi_4, \nabla_v\bar{\phi}_1) = -\int d\tilde{v}\,\tilde{v}\bar{\phi}_1 \tag{5.24}$$

Since $J(\phi_0, \bar{f}_1) = (\varphi/\rho)\,\mathscr{L}\bar{f}_1 - \Gamma\bar{f}_1$, we get

$$\Gamma^{-1}J(\phi_0, \bar{f}_1) = \frac{\varphi}{\rho}\Gamma^{-1}\mathscr{L}\bar{f}_1 - \bar{f}_1 = \frac{\varphi}{\rho}\Gamma^{-1}\mathscr{P}^{\perp}(D_t f_0) - \bar{f}_1$$

after using (5.2) to get the second equality The second term does not contribute to (5.24) since $\bar{f}_1$ is orthogonal to $M\tilde{v}$. Hence, by (5.18),

$$\int dv\,\tilde{v}\Gamma^{-1}J(\phi_0, \bar{f}_1) = \frac{\varphi}{\rho}\int d\tilde{v}\,\tilde{v}\Gamma^{-1}\left[\tilde{v}\cdot\nabla T\frac{M}{2T^2}(\tilde{v}^2 - 5T)\right] \tag{5.25}$$

the term proportional to ∇u is odd in $\tilde{v}$ and hence does not contribute to the previous expression. Now we compute

$$\begin{aligned}\mathscr{K}^{\perp}&[D_t(\varphi M_s) + \rho W\cdot\nabla_v M_s + \phi F\cdot\nabla_v M_s]\\ &= \varphi D_t^u(M_s) + \varphi\tilde{v}\cdot\nabla M_s + M_s\tilde{v}\cdot\nabla\varphi + \rho W\cdot\nabla_v M_s + \varphi F\cdot\nabla_v M_s\end{aligned} \tag{5.26}$$

We have

$$\varphi D_t^u(M_s) = \frac{M_s}{2T^2}\varphi D_t^u T(\tilde{v}^2 - 3T] - \nabla_v M_s\varphi D_t^u u$$

Since the first term is even in $\tilde{v}$, only the second term contributes to $\int dv\,\tilde{v}\Gamma_M^{-1}$ in (5.25). By using the equation for the momentum we get

$$D_t^u u = \frac{1}{\rho}[-\nabla P + \rho F + \varphi W] - \frac{\varepsilon}{\rho}\nabla\cdot\sigma \tag{5.27}$$

Moreover

$$\left[\frac{\varphi}{\rho}\nabla P - \varphi F - \frac{\varphi^2}{\rho}W + \rho W + \varphi F\right]\cdot\nabla_v M_s = \left[\frac{\varphi}{\rho}\nabla P + W\frac{\rho^2-\varphi^2}{\rho}\right]\cdot\nabla_v M_s$$

Now

$$-\frac{\varphi}{\rho}\nabla P\cdot\frac{\tilde{v}}{T}M_s + M_s\tilde{v}\cdot\nabla\varphi = \left[-\varphi\frac{1}{T}\nabla T - \frac{\varphi}{\rho}\nabla\rho + \nabla\varphi\right]\cdot M_s\tilde{v}$$
$$= \left[-\varphi\frac{1}{T}\nabla T + \rho\,\nabla\frac{\varphi}{\rho}\right]\cdot M_s\tilde{v}$$

and

$$-\varphi\frac{1}{T}\tilde{v}\cdot\nabla T M_s + \varphi\tilde{v}\cdot\nabla M_s$$
$$= \left[-\frac{1}{T}\nabla T + \left(\frac{\tilde{v}^2}{2T^2} - \frac{3}{2T}\right)\right]\varphi\tilde{v}\cdot\nabla T M_s + \frac{\varphi}{T}M_s\tilde{v}\otimes\tilde{v}\cdot\nabla u$$
$$= \varphi\,\nabla T\cdot M_s\tilde{v}\left[\frac{\tilde{v}^2}{2T^2} - \frac{5}{2T}\right] + \frac{\varphi}{T}M_s\tilde{v}\otimes\tilde{v}\cdot\nabla u \tag{5.28}$$

The last term in (5.28) does not contribute to (5.24). Since the first term in the r.h.s of (5.28) cancels out with the term in (5.25) (remember that $M = \rho M_s$) we have

$$\text{r.h.s (5.26)} = M\tilde{v}\left[\nabla\frac{\varphi}{\rho} - \frac{W}{\rho^2 T}(\rho^2 - \varphi^2) - \varepsilon\frac{\varphi}{\rho^2}\nabla\cdot\sigma\right] \tag{5.29}$$

It is now easy to check that in the computation of the l.h.s. of (5.23) the last term of (5.29) is compensated by the term $\varepsilon(\varphi/\rho^2)\,\mathscr{P}[D_t\bar{f}_1]$. Collecting terms we get

$$(M_s\chi_4, \nabla_v\bar{\phi}_1) = -D\left[\nabla\frac{\phi}{\rho} - \frac{W}{\rho^2 T}(\rho^2 - \phi^2)\right] \tag{5.30}$$

where

$$D := \int dv\; M\tilde{v}_i \Gamma^{-1}\tilde{v}_i \tag{5.31}$$

The other terms appearing in the equation for the energy are computed in the standard way:

$$\int d\tilde{v}\,\chi_4[\varphi\cdot\nabla\bar{f}_1+F_0\cdot\nabla_v\bar{f}_1+W_0\cdot\nabla_v\bar{\phi}_1]=0$$

$$\nabla\cdot\int d\tilde{v}\,\chi_4\bar{f}_1=-\nabla\cdot[\kappa\,\nabla T] \tag{5.32}$$

with

$$\kappa=\int d\tilde{v}\,\psi_2 B_i^2$$

Moreover

$$\int dv\,\chi_4 F_0\cdot\nabla_v\bar{f}_1=-\int dv\,\tilde{v}\cdot F_0\bar{f}_1=0$$

$$\int dv\,\chi_4 W_0\cdot\nabla_v\bar{\phi}_1=-\int dv\,\tilde{v}\cdot W_0\bar{\phi}_1=-W_0\cdot DQ$$

with

$$Q:=\nabla\frac{\varphi}{\rho}-\frac{1}{\rho^2 T}\,W(\rho^2-\varphi^2) \tag{5.33}$$

Therefore, the equation for the energy is

$$\tfrac{3}{2}\rho D_t T+p\,\nabla\cdot u=\varepsilon\,\nabla(\kappa\,\nabla T)-\varepsilon\sigma:\nabla u-\varepsilon W\cdot DQ \tag{5.34}$$

Finally, to get the equation for the concentration we have to exploit the condition (5.5).

$$\mathscr{K}[D_t(\varphi M+\varepsilon\bar{\phi}_1)]=\partial_t\varphi+\int dv\,v\cdot\nabla\varphi M+\varepsilon\int dv\,v\cdot\nabla\bar{\phi}_1$$

$$=\partial_t\varphi+\nabla\cdot(u\varphi)+\varepsilon\nabla\cdot\int dv\,\tilde{v}\bar{\phi}_1 \tag{5.35}$$

The last term has already been computed in (5.30). Therefore, the equation for φ is

$$\partial_t\varphi+\nabla\cdot(\varphi u)=\varepsilon\,\nabla\cdot(DQ) \tag{5.36}$$

where Q is given in (5.33).

Recalling the definitions of F_0 and W_0 we finally get the *Vlasov–Navier–Stokes equations* (VNS) for a binary mixture given by (3.5).

As for the Euler limit, the arguments in the Appendix A prove the following Proposition 5.1 which holds under the same assumptions as before Proposition 3.1: periodic boundary conditions and smoothness of the long range potential.

Proposition 5.1. Suppose that for $\varepsilon > 0$ small enough there is a solution $(\rho^\varepsilon, u^\varepsilon, T^\varepsilon, \varphi^\varepsilon)$ to the Vlasov–Navier–Stokes equations (3.5) sufficiently smooth in the time interval $[0, \bar{t}]$ independent of ε. Then there are positive constants ε_0 and C such that, for $\varepsilon < \varepsilon_0$ an unique classical solution to the system (5.14) with $m \geqslant 4$ exists and satisfies the bounds

$$\begin{aligned} \sup_{t \in [0, \bar{t}]} \|R_f(\cdot, t)\|_{\alpha, \ell, s} &\leqslant C\varepsilon \sup_{t \in [0, \bar{t}]} [\|A_f(\cdot, t)\|_{\alpha, \ell, s} + \|A_\phi(\cdot, t)\|_{\alpha, \ell, s}] \\ \sup_{t \in [0, \bar{t}]} \|R_\phi(\cdot, t)\|_{\alpha, \ell, s} &\leqslant C\varepsilon \sup_{t \in [0, \bar{t}]} [\|A_f(\cdot, t)\|_{\alpha, \ell, s} + \|A_\phi(\cdot, t)\|_{\alpha, \ell, s}] \end{aligned} \tag{5.37}$$

for any $\alpha < \bar{T}/2$, $\bar{T} \overset{\text{def}}{=} \sup_{x \in \Omega, t \in [0, \bar{t}]} T^\varepsilon(x, t)$, $\ell > 3$, $s \geqslant 2$.

Corollary 5.2. Under the assumptions of Proposition 3.1, for $\varepsilon < \varepsilon_0$ there is a smooth solution $(f_t^\varepsilon, \phi_t^\varepsilon)$ to the rescaled Vlasov–Boltzmann equation (3.1) and moreover, denoting by M_t the Maxwellian with parameters evolving according to the Vlasov–Navier–Stokes equations, it satisfies:

$$\sup_{0 \leqslant t \leqslant \bar{t}} \left[\|f_t^\varepsilon - M_t - \varepsilon f_1\|_{\alpha, \ell, s} + \left\| \phi_t^\varepsilon - \frac{\varphi_t}{\rho_t} M_t - \varepsilon \phi_1 \right\|_{\alpha, \ell, s} \leqslant C\varepsilon^2 \right]$$

6. INCOMPRESSIBLE NAVIER–STOKES LIMIT

In this Section we consider a different scaling limit such that one can get hydrodynamic equations with dissipative terms of order 1 instead of order ε as in the previous section. We choose also in this case $\gamma = \varepsilon$ but we use the parabolic space time scaling

$$t \to \varepsilon^{-2} t, \qquad x \to \varepsilon^{-1} x$$

After rescaling, Eqs. (2.11) become:

$$\begin{aligned} \partial_t f + \varepsilon^{-1} v \cdot \nabla_x f + \varepsilon^{-1} F \cdot \nabla_v f + \varepsilon^{-1} W \cdot \nabla_v \phi &= \varepsilon^{-2} J(f, f) \\ \partial_t \phi + \varepsilon^{-1} v \cdot \nabla_x \phi + \varepsilon^{-1} F \cdot \nabla_v \phi + \varepsilon^{-1} W \cdot \nabla_v f &= \varepsilon^{-2} J(\phi, f) \end{aligned} \tag{6.1}$$

with F and W given by (3.2).

We shall solve (6.1) as in the Euler case in terms of a truncated Hilbert expansion of the form

$$f = \sum_{n=0}^{K} \varepsilon^n f_n + \varepsilon^m R_f$$
$$\phi = \sum_{n=0}^{K} \varepsilon^n \Phi_n + \varepsilon^m R_\Phi \tag{6.2}$$

with suitably chosen positive integers K and m, but in this case we choose in a different way the terms of order 0 in the expansion.

$$f_0 = M_0, \qquad \phi_0 = 0$$

where M_0 is a Maxwellian with parameters $\bar{\rho}$ and $\bar{T}$ some fixed constants and $u = 0$. This implies the vanishing of the forces at the lowest order: $W_0 = F_0 = 0$. We remark that choosing the first order term in the expansion to be a global Maxwellian is essential to the incompressible limit set up. The choice $\phi_0 = 0$ is made for simplifying the computations. By plugging (6.2) in the rescaled Boltzmann equations (6.1), one easily obtains the conditions for the higher order terms in the expansions:

$$\mathscr{L} f_1 = 0 \tag{6.3}$$

$$\Gamma \phi_1 = 0 \tag{6.4}$$

for $0 \leqslant n \leqslant K-1$:

$$\mathscr{L} f_{n+2} + \sum_{\substack{(h,h'):\, h,h' \geqslant 1 \\ h+h'=n+2}} Q(f_h, f_{h'}) = \sum_{\substack{(h,h'):\, h>0,\, h' \geqslant 0 \\ h+h'=n+1}} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}] + v \cdot \nabla_x f_{n+1} + \partial_t f_n \tag{6.5}$$

$$\Gamma \phi_{n+2} + \sum_{\substack{(h,h'):\, h,h' \geqslant 1 \\ h+h'=n+2}} J(\phi_h, f_{h'}) = \sum_{\substack{(h,h'):\, h,h' \geqslant 0 \\ h+h'=n+1}} [F_h \cdot \nabla_v \phi_{h'} + W_h \cdot \nabla_v f_{h'}] + v \cdot \nabla_x \phi_{n+1} + \partial_t \phi_n \tag{6.6}$$

Moreover

$$\begin{aligned}
&\partial_t R_f + \varepsilon^{-1}[v \cdot R_f + F \cdot \nabla_v R_f + W \cdot \nabla_v R_\phi] \\
&\qquad = \varepsilon^{-2} \mathscr{L} R_f + \varepsilon^{-1} \mathscr{L}^{(1)} R_f + \mathscr{L}^{(2)} R_f + \varepsilon^{m-2}[J(R_f, R_f) + A_f] \\
&\partial_t R_\phi + \varepsilon^{-1}[v \cdot R_\phi + F \cdot \nabla_v R_\phi + W \cdot \nabla_v R_f] \\
&\qquad = \varepsilon^{-2} \Gamma R_\phi + \varepsilon^{-1} \Gamma^{(1)} R_\phi + \varepsilon^{-1} \tilde{\Theta}^{(1)} R_f + \Gamma^{(2)} R_\phi + \tilde{\Theta}^{(2)} R_f \\
&\qquad\quad + \varepsilon^{m-2}[J(R_\phi, R_f) + A_\phi]
\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}
&\mathscr{L}^{(1)} g = J(f_1, g) + J(g, f_1), \qquad \mathscr{L}^{(2)} g = \sum_{h=2}^{K} \varepsilon^{h-2}[J(f_h, g) + J(g, f_h)] \\
&\tilde{\Theta}^{(1)} g = J(\phi_1, g), \qquad \tilde{\Theta}^{(2)} g = J\left(\sum_{n=2}^{K} \varepsilon^{n-2} \phi_n, g\right) \\
&\Gamma^{(1)} g = J(g, f_1), \qquad \Gamma^{(2)} g = \sum_{h=2}^{K} \varepsilon^{h-2} J(g, f_h)
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
A_f = \varepsilon^{K-2m+2} \Bigg(& \sum_{\substack{(h, h') : h, h' \geqslant 1 \\ h+h' > K+2}} \varepsilon^{h+h'-K-2} Q(f_h, f_{h'}) \\
& - \sum_{\substack{(h, h') : h, h' \geqslant 0 \\ h+h' > K+1}} \varepsilon^{h+h'-K-1} [F_h \cdot \nabla_v f_{h'} + W_h \cdot \nabla_v \phi_{h'}] \\
& - \partial_t f_{K-1} - v \cdot \nabla_x f_K - \varepsilon\, \partial f_K \Bigg) \\
A_\phi = \varepsilon^{K-2m+1} \Bigg(& \sum_{\substack{(h, h') : h, h' \geqslant 1 \\ h+h' > K+2}} \varepsilon^{h+h'-K-2} J(\phi_h, f_{h'}) \\
& - \sum_{\substack{(h, h') : h, h' \geqslant 0 \\ h+h' > K+1}} \varepsilon^{h+h'-K-1} [W_h \cdot \nabla_v f_{h'} + F_h \cdot \nabla_v \phi_{h'}] \\
& - \partial_t \phi_{K-1} - v \cdot \nabla_x \phi_K - \varepsilon\, \partial \phi_K \Bigg)
\end{aligned} \tag{6.9}$$

and F and W are given by

$$F = \sum_{n=1}^{K} \varepsilon^n F_n + \varepsilon^m \mathbf{K} \circledast R_f$$
$$W = \sum_{n=1}^{K} \varepsilon^n W_n + \varepsilon^m \mathbf{K} \circledast R_\phi \tag{6.10}$$

with $F_n = \mathbf{K} \circledast f_n$, $W_n = \mathbf{K} \circledast \phi_n$.

We now find the expressions of the first terms in the expansions f_1, ϕ_1, f_2, ϕ_2. From (6.4) we get

$$\phi_1 = \varphi_1 M_0$$

and from (6.3) we have

$$f_1 = M_0(v)\left(\rho_1 + u_1 \cdot \frac{v}{\overline{T}} + \theta_1 \frac{v^2 - 3\overline{T}}{\overline{T}^2}\right)$$

where ρ_1, u_1, θ_1 are to be determined as functions of x, t. By (6.5) with $n = 0$ we obtain

$$\mathscr{P}(v \cdot \nabla_x f_1 + F_1 \cdot \nabla_v f_0) = 0$$

But

$$\mathscr{P}[v \cdot \nabla_x f_1] = M_0\left[\left(1 + \frac{v^2 - 3\overline{T}}{2\overline{T}^2}\right)\nabla_x \cdot u_1 + \frac{v}{\overline{T}} \cdot \nabla_x(\overline{T}\rho_1 + \bar{\rho}\theta_1)\right]$$

while

$$\mathscr{P}[F_1 \cdot \nabla_v f_0] = -M_0 \frac{v}{\overline{T}} \cdot F_1$$

Hence we find the conditions

$$\nabla_x \cdot u_1 = 0, \qquad \nabla_x\left[\rho_1 + \theta_1 + \int dy\, U(|x-y|)\, \rho_1(y)\right] = 0 \tag{6.11}$$

On the other hand,

$$\mathscr{P}^{\perp}[F_1 \cdot \nabla_v f_0] = 0$$

so

$$f_2 = \mathscr{L}^{-1}[\mathscr{P}^{\perp} v \cdot \nabla_x f_1] - \mathscr{L}^{-1} J(f_1, f_1) + \hat{f}_2$$

with $\hat{f}_2 \in \text{Null } \mathscr{L}$.

Therefore f_2 has the usual expression

$$f_2 = \frac{1}{2} \sum_{i,j=1}^{3} A_{i,j}[u_{1,i} u_{1,j} - \psi_1 \, \partial_i u_{1,j}] + \sum_{i=1}^{3} B_i[\theta_1 u_{1,i} - \psi_2 \, \partial_i \theta_1] + \frac{1}{2} M_0 \theta_1^2 \mathscr{P}^{\perp} \left[\left(\frac{v^2 - 3}{2} \right)^2 \right] \tag{6.12}$$

From (6.6) with $n = 0$ we get the expression for ϕ_2:

$$\phi_2 = \Gamma^{-1}[v \cdot \nabla_x \phi_1 + W_0 \nabla_v f_0 - J(\phi_1, f_1)] + \hat{\phi}_2$$

with $\hat{\phi}_2 \in \text{Null } \Gamma$.

Moreover, by (6.3)

$$\Gamma^{-1} J(\phi_1, f_1) = -\varphi_1 \Gamma^{-1} \Gamma f_1 = -\varphi_1 f_1$$

Hence

$$\phi_2 = \varphi_1 f_1 + \left[\nabla_x \varphi_1 - \frac{1}{\bar{T}} W_1 \right] \cdot \Gamma^{-1}[v M_0] + \hat{\phi}_2$$

From (6.5) with $n = 1$ we get

$$\partial_t u_1 + u_1 \cdot \nabla_x u_1 = -\nabla_x p + F_2 + \nu \, \Delta_x u_1 + \rho_1 F_1 + \varphi_1 W_1$$

and

$$\tfrac{5}{2}[\partial_t \theta_1 + u_1 \cdot \nabla_x \theta_1] = u_1 \cdot F_1 + \kappa \, \Delta_x \theta_1$$

since $F_2 = \nabla_x G$, with $G(x) = \int dy \, U(|x - y|) \, \rho_2(y)$, putting $\bar{p} = p - G$, the previous equation reduces to

$$\partial_t u_1 + u_1 \cdot \nabla_x u_1 = -\nabla_x \bar{p} + \nu \, \Delta_x u_1 + \rho_1 F_1 + \varphi_1 W_1$$

which is the usual incompressible Navier–Stokes equation with the self-consistent force

$$\rho_1 F_1 + \varphi_1 W_1 = -\rho_1 \nabla_x \int dy\ U(|x-y|)\ \rho_1(y) + \varphi_1 \nabla_x \int dy\ U(|x-y|)\ \varphi_1(y)$$

Finally from (6.4) with $n=1$ we get the equation for φ_1

$$\partial_t \varphi_1 + u_1 \cdot \nabla_x \varphi_1 = D \left[\frac{1}{\bar{\rho}} \Delta_x \varphi_1 - \frac{1}{\bar{\bar{T}}} \Delta_x \int dy\ U(|x-y|)\ \varphi_1(y)] \right]$$

with

$$D = -\int dv\ v \cdot \Gamma^{-1}(vM) \tag{6.13}$$

Summarizing, dropping the index 1, the set of equation for ρ, u, θ, ϕ, p is:

$$\begin{gathered}
\partial_t u + u \cdot \nabla_x u = -\nabla_x p + v\, \Delta_x u + \rho F + \varphi W \\
\frac{5}{2} [\partial_t \theta + u \cdot \nabla_x \theta] = u \cdot F + k\, \Delta_x \theta \\
\partial_t \varphi + u \cdot \nabla_x \varphi = D \left[\frac{1}{\bar{\rho}} \Delta_x \varphi - \frac{1}{\bar{\bar{T}}} \Delta_x \int dy\ U(|x-y|)\ \varphi(y)] \right] \\
F = -\nabla_x \int dy\ U(|x-y|)\ \rho(y) \\
W = \nabla_x \int dy\ U(|x-y|)\ \varphi(y) \\
\nabla_x \left[\rho + \theta + \int dy\ U(|x-y|)\ \rho(y) \right] = 0 \\
\nabla_x \cdot u = 0
\end{gathered} \tag{6.14}$$

The equation for φ is linear unlike the one we get in the VNS equations, but there is still a non linear term in φ in the momentum equation. The equation for θ which corresponds to the deviation in the temperature decouples from the rest. In fact, if we consider a solution to the previous equation with an initial datum $\rho = const$, $\theta = const$ such conditions persist in time and u and φ have to solve the simplified set of equations

$$\begin{aligned}
&\partial_t u + u \cdot \nabla_x u = -\nabla_x p + \nu\, \Delta_x u + \varphi W \\
&\partial_t \varphi + u \cdot \nabla_x \varphi = D\left[\frac{1}{\bar\rho} \Delta_x \varphi - \frac{1}{\bar T} \Delta_x \int dy\, U(|x-y|)\, \varphi(y)]\right] \\
&W = \nabla_x \int dy\, U(|x-y|)\, \varphi(y) \\
&\nabla_x \cdot u = 0
\end{aligned} \tag{6.15}$$

In Appendix B there is the proof of the following proposition:

Proposition 6.1. Suppose that for $\varepsilon > 0$ small enough there is a solution (ρ, u, T, φ) to the incompressible Navier–Stokes equations (6.14) sufficiently smooth in the time interval $[0, \bar t\,]$ independent of ε. Then there are positive constants ε_0 and C such that, for $\varepsilon < \varepsilon_0$ a unique classical solution to the system (6.7) exists and satisfies the bounds

$$\begin{aligned}
\sup_{t \in [0, \bar t\,]} \|R_f(\cdot, t)\|_{\alpha, \ell, s} \leqslant C\varepsilon \sup_{t \in [0, \bar t\,]} [\|A_f(\cdot, t)\|_{\alpha, \ell, s} + \|A_\phi(\cdot, t)\|_{\alpha, \ell, s}] \\
\sup_{t \in [0, \bar t\,]} \|R_\phi(\cdot, t)\|_{\alpha, \ell, s} \leqslant C\varepsilon \sup_{t \in [0, \bar t\,]} [\|A_f(\cdot, t)\|_{\alpha, \ell, s} + \|A_\phi(\cdot, t)\|_{\alpha, \ell, s}]
\end{aligned} \tag{6.16}$$

for any $\alpha < \bar T/2$, $\ell > 3$, $s < m$.

Corollary 6.2. Under the assumptions of Proposition 3.1, there is for $\varepsilon < \varepsilon_0$ a smooth solution $(f_t^\varepsilon, \phi_t^\varepsilon)$ to the rescaled Vlasov–Boltzmann equation (6.1) and moreover, denoting by M_t the Maxwellian with parameters evolving according to the incompressible Navier–Stokes equations (6.14), it satisfies:

$$\sup_{0 \leqslant t \leqslant \bar t} \|f_t^\varepsilon - M_0 - \varepsilon f_1\|_{\alpha, \ell, s} + \|\phi_t^\varepsilon - M_0 - \varepsilon \phi_1\|_{\alpha, \ell, s} \leqslant C\varepsilon^2]$$

APPENDIX A

We present a sketch of the proof of Proposition 3.1 in the case of hard spheres, where $\nu(v) \approx |v|$ for large v's. The extension to more general cross sections is possible along the lines proposed in [DE], but we do not discuss it. The smoothness and decay properties of the expansion terms are obtained by now standard methods ([Ca80, DEL, ELM94, ELM95, ELM98, ELM99]) which allow to prove the following

Theorem A.1. Given $\varepsilon > 0$, assume that there exists a sufficiently smooth solution of the Vlasov–Euler equations (4.24) and (4.25) in the time interval $[0, \bar{t}]$. Then for any $j > 3$, $s < s_0$ and $0 < \alpha < T^* = \sup_{(x,t) \in \Omega \times [0,\bar{t}]} T(x,t)$ there is a constant $c > 0$ such that the terms in the expansion f_i, ϕ_i, $i = 1,..., K$, with $K = 2m$, solutions of the Eqs. (5.8)–(5.7)

$$\|f_i\|_{\alpha, j, s} \leqslant c, \qquad \|\phi_i\|_{\alpha, j, s} \leqslant c \tag{A.1}$$

$$\|\partial_v f_i\|_{\alpha, j, 2} \leqslant c, \qquad \|\partial_v \phi_i\|_{\alpha, j, 2} \leqslant c \tag{A.2}$$

We need bounds on the remainders R_f and R_ϕ.
We set

$$R^r \equiv R^{(1)} = R_f - R_\phi, \qquad R^b \equiv R^{(2)} = R_f + R_\phi \tag{A.3}$$

The reason is that terms like $W \cdot \nabla_v R_\phi$ and $F \cdot \nabla_v R_f$ in (4.26) are not well suited when some force term is present to represent the solutions of the equations in terms of characteristics, which is essential in the method we are going to use. The equations for the new variables are:

$$\begin{aligned} &D_t R^{(1)} + F^{(1)} \cdot \nabla_v R^{(1)} \\ &\quad = \varepsilon^{-1}[\rho^{(1)} \bar{\mathscr{L}} R^{(1)} + \rho^{(1)} \bar{\Theta} R^{(2)} + \rho^{(2)} \bar{\Gamma} R^{(1)}] + \mathscr{L}_1 R^{(1)} \\ &\qquad + \Gamma_2 R^{(1)} + \Theta_1 R^{(2)} + \varepsilon^{m-1}[J(R^{(1)}, R^{(1)}) + J(R^{(1)}, R^{(2)}) + A^{(1)}] \\ &D_t R^{(2)} + F^{(2)} \cdot \nabla_v R^{(2)} \\ &\quad = \varepsilon^{-1}[\rho^{(2)} \bar{\mathscr{L}} R^{(2)} + \rho^{(2)} \bar{\Theta} R^{(1)} + \rho^{(1)} \bar{\Gamma} R^{(2)}] + \mathscr{L}_2 R^{(2)} \\ &\qquad + \Gamma_1 R^{(2)} + \Theta_2 R^{(1)} + \varepsilon^{m-1}[J(R^{(2)}, R^{(2)}) + J(R^{(2)}, R^{(1)}) + A^{(2)}] \end{aligned} \tag{A.4}$$

where $\bar{M}$ is the Maxwellian M with $\rho = 1$ and

$$f_j = \tfrac{1}{2}[f_j^{(1)} + f_j^{(2)}], \qquad \phi_j = \tfrac{1}{2}[-f_j^{(1)} + f_j^{(2)}]$$

$$\bar{\mathscr{L}} h = J(\bar{M}, h) + J(h, \bar{M})$$

$$\bar{\Theta} h = J(\bar{M}, h), \qquad \bar{\Gamma} h = J(h, \bar{M})$$

$$\mathscr{L}_i h = J\left(\sum_{j=1}^{K} \varepsilon^{j-1} f_j^{(i)}, h\right) + J\left(h, \sum_{j=1}^{K} \varepsilon^{j-1} f_j^{(i)}\right)$$

$$\Gamma_i h = J\left(h, \sum_{j=1}^{K} \varepsilon^{j-1} f_j^{(i)}\right), \qquad \Theta_i h = J\left(\sum_{j=1}^{K} \varepsilon^{j-1} f_j^{(i)}, h\right)$$

$$A^{(1)} = A_f + A_\phi, \qquad A^{(2)} = A_f - A_\phi$$

Following Caflisch [Ca80] we now decompose the remainders in low and high velocity parts, by looking for solutions to Eqs. (A.4) in the form

$$R^{(1)} = \sqrt{\rho^{(1)} \bar{M}}\, g^{(1)} + \sqrt{M^*}\, h^{(1)}, \qquad R^{(2)} = \sqrt{\rho^{(2)} \bar{M}}\, g^{(2)} + \sqrt{M^*}\, h^{(2)}$$

M^* is a global Maxwellian with a temperature T^*. We have

$$\begin{aligned} &D_t g^{(1)} + F^{(1)} \cdot \nabla g^{(1)} \\ &\quad = \varepsilon^{-1} [\rho^{(1)} \bar{L} g^{(1)} + \sqrt{\rho^{(1)}}\sqrt{\rho^{(2)}}\, \bar{T} g^{(2)} + \rho^{(2)} \bar{G} g^{(1)}] \\ &\qquad + \varepsilon^{-1} \chi \sigma^{-1} \left[\sqrt{\rho^{(1)}} (K^* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\sqrt{\rho^{(1)}}} K_G^* h^{(1)} \right] \end{aligned} \tag{A.5}$$

$$\begin{aligned} &D_t h^{(1)} + F^{(1)} \cdot \nabla h^{(1)} \\ &\quad = \sigma [\mu^{(1)} + F^{(1)} \cdot \mu'^{(1)}] \sqrt{\rho^{(1)}}\, g^{(1)} + F^{(1)} \cdot \mu'_* h^{(1)} \\ &\qquad + \varepsilon^{-1} \rho^{1)} \left[-\nu + \bar{\chi} (K^* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\rho^{(1)}} (-\nu_G + K_G^* h^{(1)}) \right] \\ &\qquad + L_1 (\sigma \sqrt{\rho^{(1)}}\, g^{(1)} + h^{(1)}) + G_2 (\sigma \sqrt{\rho^{(1)}}\, g^{(1)} + h^{(1)}) \\ &\qquad + T_1 (\sigma \sqrt{\rho^{(2)}}\, g^{(2)} + h^{(2)}) \\ &\qquad + \varepsilon^{m-1} [\nu^* Q^* (\sigma \rho^{(1)} g^{(1)} + h^{(1)}, \sigma \rho^{(1)} g^{(1)} + h^{(1)}) \\ &\qquad + \nu^* Q^* (\sigma \rho^{(1)} g^{(1)} + h^{(1)}, \sigma \rho^{(2)} g^{(2)} + h^{(2)}) + A^{(1)}] \end{aligned}$$

The equation for $g^{(2)}$, $h^{(2)}$ is obtained by the exchange $1 \to 2$ where

$$\chi(v) = \begin{cases} 1, & |v| \leqslant \gamma \\ 0, & \text{otherwise} \end{cases} \qquad \bar{\chi} = 1 - \chi \tag{A.6}$$

$$\begin{aligned} &\mu^{(i)} = \frac{1}{2} D_t (\log \bar{M}), \qquad \mu'^{(i)} = \frac{1}{2} \rho^{(i)} \nabla_v \log \bar{M}, \qquad i = 1, 2 \\ &\mu'_* = \frac{1}{2} \nabla_v \log M_*, \qquad \sigma = \sqrt{\frac{\bar{M}}{M^*}} \end{aligned} \tag{A.7}$$

$$\bar{L}f = \frac{1}{\sqrt{\bar{M}}}\bar{\mathscr{L}}\sqrt{\bar{M}}\,f) = (-\nu + K)\,f$$

$$L^*f = \frac{1}{\sqrt{M^*}}\mathscr{L}\sqrt{M^*}\,f) = -\nu^* + K^*f$$

$$\bar{T}f = \frac{1}{\sqrt{\bar{M}}}\bar{\Theta}\sqrt{\bar{M}}\,f$$

$$\bar{G}f = \frac{1}{\sqrt{\bar{M}}}\bar{\Gamma}\sqrt{\bar{M}}\,f = (-\nu_G + \bar{K}_G)\,f$$

$$K_T^* = \frac{1}{\sqrt{M^*}}\bar{\Theta}\sqrt{M^*}\,f \tag{A.8}$$

$$G^*f = \frac{1}{\sqrt{M^*}}\bar{\Gamma}\sqrt{M^*}\,f = (-\nu_G^* + K_G^*)\,f$$

$$G_i f = \frac{1}{\sqrt{M^*}}\Gamma_i\sqrt{M^*}\,f, \qquad i = 1, 2$$

$$L_i f = \frac{1}{\sqrt{M^*}}\mathscr{L}_i\sqrt{M^*}\,f, \qquad i = 1, 2$$

$$T_i f = \frac{1}{\sqrt{M^*}}\Theta_i\sqrt{M^*}\,f, \qquad i = 1, 2$$

$$\begin{aligned} \nu^*Q^*(f, \ell) &= \frac{1}{\sqrt{M^*}}Q(\sqrt{M^*}\,f, \sqrt{M^*}\,\ell) \\ \nu^*J^*(f, \ell) &= \frac{1}{\sqrt{M^*}}J(\sqrt{M^*}\,f, \sqrt{M^*}\,\ell) \end{aligned} \tag{A.9}$$

Linear Problem

We solve first the linear problem for g_i, h_i, $i = 1, 2$, assuming that $F^{(i)}$ are given functions such that

$$\|F^{(i)}\|_\infty + \|\nabla_x F^{(i)}\|_\infty < \alpha_F \tag{A.10}$$

We consider the linear system

$$
\begin{aligned}
&D_t g^{(1)} + F^{(1)} \cdot \nabla g^{(1)} \\
&\quad = \varepsilon^{-1}[\rho^{(1)} \bar{L} g^{(1)} + \sqrt{\rho^{(1)}}\sqrt{\rho^{(2)}}\, \bar{T} g^{(2)} + \rho^{(2)} \bar{G} g^{(1)}] \\
&\quad\quad + \varepsilon^{-1} \chi \sigma^{-1} \left[\sqrt{\rho^{(1)}} (K^* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\sqrt{\rho^{(1)}}} K_G^* h^{(1)} \right] \qquad \text{(A.11)} \\
&D_t h^{(1)} + F^{(1)} \cdot \nabla h^{(1)} \\
&\quad = \sigma[\mu^{(1)} + F^{(1)} \cdot \mu'^{(1)}] \sqrt{\rho^{(1)}}\, g^{(1)} + F^{(1)} \cdot \mu'_* h^{(1)} \\
&\quad\quad + \varepsilon^{-1} \rho^{1)} \left[-\nu + \bar{\chi} (K^* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\rho^{(1)}} (-\nu_G + K_G^* h^{(1)}) \right] \\
&\quad\quad + L_1(\sigma \sqrt{\rho^{(1)}}\, g^{(1)} + h^{(1)}) + G_2(\sigma \sqrt{\rho^{(1)}}\, g^{(1)} + h^{(1)}) \\
&\quad\quad + T_1(\sigma \sqrt{\rho^{(2)}}\, g^{(2)} + h^{(2)}) + \varepsilon^{m-1} D^{(1)}
\end{aligned}
$$

and the equation for $g^{(2)}$, $h^{(2)}$ obtained by the exchange $1 \to 2$. Here $F^{(1)}$ has to be considered as a given force.

We use the integral form of (A.11) [ELM98]:

$$
g^{(i)}(t, x, v) = \int_{t^-}^{t} ds\, H^{(i)}(s, \varphi_{s-t}^{(i)}(x, v)) \exp\left\{ -\int_s^t ds' \frac{1}{\varepsilon} \tilde{\nu}^{(i)}(\varphi_{s'-t}^{(i)}(x, v)) \right\} \qquad \text{(A.12)}
$$

where $\tilde{\nu}^{(i)} = \rho^{(i)} \nu + \rho^{(j)} \nu_G$, $\varphi_t^{(i)}(x, v)$ the characteristics of the equation

$$
\partial_t f + v \cdot \nabla_x f + F^{(i)} \cdot \nabla_v f = 0 \qquad \text{(A.13)}
$$

and

$$
\begin{aligned}
H^{(1)} &= \varepsilon^{-1}[\rho^{(1)} K g^{(1)} + \sqrt{\rho^{(1)}}\sqrt{\rho^{(2)}}\, \bar{T} g^{(2)} + \rho^{(2)} K_G g^{(1)}] \\
&\quad + \varepsilon^{-1} \chi \sigma^{-1} \left[\sqrt{\rho^{(1)}} (K^* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\sqrt{\rho^{(1)}}} K_G^* h^{(1)} \right] \qquad \text{(A.14)}
\end{aligned}
$$

and $H^{(2)}$ is given by the same expression after the exchange $1 \to 2$.

$$
h^{(i)}(t, x, v) = \int_{t^-}^{t} ds\, H'^{(i)}(s, \varphi(i)_{s-t}(x, v)) \exp\left\{ -\int_s^t ds' \frac{1}{\varepsilon} \hat{\nu}^{(i)}(\varphi(i)_{s'-t}(x, v)) \right\} \qquad \text{(A.15)}
$$

with

$$
\hat{\nu}^{(i)} = \tilde{\nu}^{(i)} - \varepsilon \mu'_* \cdot F^{(i)}
$$

(which is positive for ε sufficiently small, depending on α_F, since $\tilde{v}^{(i)}$ grow linearly at high velocities) and

$$
\begin{aligned}
H'^{(1)} = \varepsilon^{-1}\rho^{(1)} & \left[\bar{\chi}(K_* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\rho^{(1)}} (K_G^* h^{(1)}) \right] \\
& + L^{(1)}(\sigma \sqrt{\rho^{(1)}}\, g^{(1)} + h^{(1)}) + G^{(2)}(\sigma \sqrt{\rho^{(1)}}\, g^{(1)} + h^{(1)}) \\
& + T^{(1)}(\sigma \sqrt{\rho^{(2)}}\, g^{(2)} + h^{(2)}) \\
& + \sigma[\mu^{(1)} + F^{(1)}\mu'^{(1)}] \sqrt{\rho^{(1)}}\, g^{(1)} + \varepsilon^{m-1} D^{(1)}
\end{aligned} \tag{A.16}
$$

We do not write explicitly the equations for $g^{(2)}$, $h^{(2)}$ in integral form. In the following we use the compact notation: $g = \{g^{(1)}, g^{(2)}\}$ and $h = \{h^{(1)}, h^{(2)}\}$. Below we use the notation $\|\cdot\|_{\ell,s} = \|\cdot\|_{0.\ell,s}$ and $\|\cdot\|_{\ell} = \|\cdot\|_{0.\ell,0}$. Generalizing the method by Caflisch [Ca80] to our case, we get bounds for the norms $\|\cdot\|_{\ell}$ of $g^{(i)}$, $h^{(i)}$ in terms of the L_2 norm of $g^{(i)}$ in the form

$$
\begin{aligned}
&\|h\|_r \leqslant \varepsilon(1+\alpha_F)\, \|g\|_{L_2} + \varepsilon^m\, |D|_{r-1} \\
&\|g\|_r \leqslant \|g\|_{L_2} + \varepsilon^{m+1}\, |D|_r
\end{aligned} \tag{A.17}
$$

provided that ε, ε_0 for some suitable ε_0 positive and finite for any finite α_F. To conclude the argument we need a bound for $\|g\|_{L_2} = \sum_{i=1}^{2} \|g^{(i)}\|_{L_2}$ in terms of the L_2 norm of D. This last step is not standard so that we give a sketch of the proof.

To estimate $\|g\|_{L_2}$, we multiply the first equation in (A.11) by $g^{(i)}$, $i, j = 1, 2$, $i \neq j$, respectively, integrate over x, v and finally sum over $i = 1, 2$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} & [\, \|g^{(1)}\|_{L_2}^2 + \|g^{(2)}\|_{L_2}^2] \\
&= \varepsilon^{-1}[\langle \sqrt{\rho^{(1)}}\, g^{(1)}, L \sqrt{\rho^{(1)}}\, g^{(1)} \rangle + \langle \sqrt{\rho^{(2)}}\, g^{(2)}, L \sqrt{\rho^{(2)}}\, g_2 \rangle] \\
&\quad + \varepsilon^{-1}[(\langle \sqrt{\rho^{(2)}}\, g^{(1)}, \bar{M}^{-1/2} J(\bar{M}, \sqrt{\bar{M}} \sqrt{\rho^{(1)}}\, g^{(2)})\rangle \\
&\quad + \langle \sqrt{\rho^{(1)}}\, g^{(2)}, \bar{M}^{-1/2} J(\bar{M}, \sqrt{\bar{M}} \sqrt{\rho^{(2)}}\, g^{(1)} \rangle) \\
&\quad + \langle \sqrt{\rho^{(2)}}\, g^{(1)}, \bar{M}^{-1/2} J(\sqrt{\bar{M}} \sqrt{\rho^{(2)}}\, g^{(1)}, \bar{M})\rangle \\
&\quad + \langle \sqrt{\rho^{(1)}}\, g^{(2)}, \bar{M}^{-1/2} J(\sqrt{\bar{M}} \sqrt{\rho^{(1)}}\, g^{(2)}, \bar{M})\rangle] \\
&\quad + \varepsilon^{-1} \left\langle \chi\sigma^{-1} \left[\sqrt{\rho^{(1)}}(K^* h^{(1)} + K_T^* h^{(2)}) + \frac{\rho^{(2)}}{\sqrt{\rho^{(1)}}} K_G^* h^{(1)} \right], g^{(1)} \right\rangle \\
&\quad + \varepsilon^{-1} \left\langle \chi\sigma^{-1} \left[\sqrt{\rho^{(2)}}(K^* h^{(2)} + K_T^* h^{(1)}) + \frac{\rho^{(1)}}{\sqrt{\rho^{(2)}}} K_G^* h^{(2)} \right], g^{(2)} \right\rangle
\end{aligned} \tag{A.18}
$$

Here $\langle f, g\rangle$ denotes the $L_2(\Omega \times \mathbb{R}^3)$ scalar product. The operator L is symmetric with respect to $\langle \cdot, g \cdot\rangle$. The terms in the first square brackets are non positive by the non positivity of the operator L. It is easy to see, by using the symmetry properties of the Boltzmann kernel [CC], that also the contribution coming from the terms in the second square bracket are non positive. Therefore we have

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{L_2} \leqslant C\varepsilon^{-1}\|h^{(2)}\|_r \cdot \|g\|_{L_2} + \beta\, \|g\|_2\, L^2_2 \tag{A.19}$$

with $r \geqslant 3$. The final estimate is

$$\|g\|_{L_2} \leqslant C\varepsilon^{m-1}\|v^{-1}D\|_r$$

The same kind of arguments provides the bound for the derivatives of g with respect to x. Because of the force terms the argument differs from the one given in [Ca80] in the fact that we have to control at the same time the derivatives with respect to velocity and space. We sketch the proof for the derivatives of g. Differentiating the first equation in (A.11) we get two coupled equations for $\nabla_v g$ and ∇g

$$\partial_t \nabla g^{(i)} + (v \cdot \nabla)\nabla g^{(i)} + \nabla F^{(i)} \cdot \nabla_v g^{(i)} + F^{(i)} \cdot \nabla_v(\nabla g^{(i)}) = \nabla N^{(i)}(g, h)$$

$$\partial_t \nabla_v g^{(i)} + v \cdot \nabla(\nabla_v g^{(i)}) + \nabla g^{(i)} + F^{(i)} \cdot \nabla_v(\nabla_v g^{(i)}) = \nabla_v N^{(i)}(g, h)$$

where $N(g, h)$ is the r.h.s. of (A.11) for g. Proceeding as before in getting (A.18) we obtain

$$\frac{d}{dt}(\|\nabla g\|_{L_2} + \|\nabla_v g\|_{L_2}) \leqslant c\alpha_F(\|\nabla g\|_{L_2} + \|\nabla_v g\|_{L_2}) + \|\nabla N\|_{L_2} + \|\nabla_v N\|_{L_2}$$

The derivatives of N with respect to velocity can be estimated by the methods in [ELM94], where it is proven the identity

$$\frac{\partial}{\partial v} Q(f, g) = Q\left(f, \frac{\partial g}{\partial v}\right) + Q\left(g, \frac{\partial f}{\partial v}\right)$$

where $\partial/\partial v$ stands for the partial derivative with respect to any of the components of v. The final result is

Lemma A.2. There is an $\varepsilon_0 > 0$ finite for each finite α_F such that any solution to the linear problem (A.11), with D and F_i given, satisfies for $j > 3$, $s \leqslant 3$ and any $0 < \varepsilon \leqslant \varepsilon_0$

$$\|g^{(i)}\|_{j,s} + \|\partial_v g^{(i)}\|_{j,s-1}$$
$$\leqslant C(1+\alpha_F)\,\varepsilon^{m-s}[\,\|\tilde{\nu}^{-1}D^{(i)}\|_{j+2,s} + \|\tilde{\nu}^{-1}\,\partial_v D^{(i)}\|_{j+2,s}] \qquad \text{(A.20)}$$

$$\|h^{(i)}\|_{j,s} + \|\partial_v h^{(i)}\|_{j,s-1}$$
$$\leqslant C(1+\alpha_F)\,\varepsilon^{m-s+1}[\,\|\tilde{\nu}^{-1}D^{(i)}\|_{j,s} + \|\tilde{\nu}^{-1}\,\partial_v D^{(i)}\|_{j,s}] \qquad \text{(A.21)}$$

Non-linear problem and fixed point argument.

Non-Linear Problem and Fixed Point Argument

The nonlinear equations (A.5) are solved by a fixed point method. This method works if there is some small parameter in front of the non linear terms. In (A.5) there are two kinds of non linear terms: the usual Boltzmann non linear term, which is multiplied by a power of ε and the Vlasov term involving the forces which gives rise to linear and quadratic terms in the remainders: in fact the forces are given by expressions of the type

$$F^{(1)} = \mathbf{K} \circledast f^{(2)} = \left(K \circledast \left[\sum_{n=0}^{K} \varepsilon^n f_n^{(2)} \right] \right) + \varepsilon^m \mathbf{K} \circledast R^{(2)} \qquad \text{(A.22)}$$

Hence the non linear term due to the force is small and we can apply the recursive argument.

The Boltzmann terms are dealt with as in Caflisch [Ca80]. The control of the force term requires the boundedness of the gradient of the Kac potential. The result is

Theorem A.3. There is an $\varepsilon_0 > 0$ such that the remainders satisfy for $j > 3$, $z = s-(d-1)$ with $s < m$ and any $0 < \varepsilon \leqslant \varepsilon_0$

$$\|g^{(i)}\|_{j,z} \leqslant c\varepsilon^{m-s+1}, \qquad \|h^{(i)}\|_{j,z} \leqslant c\varepsilon_{m-s+1}, \qquad i=1,2 \qquad \text{(A.23)}$$

Proof. Let $R_k^{(i)}$ be the solution of (A.4) with force

$$F_k^{(i)} = \left(\mathbf{K} \circledast \left[\sum_{n=0}^{K} \varepsilon^n f_n^{(j)} \right] \right) + \varepsilon^m (\mathbf{K} \circledast R_{k-1}^{(j)})$$
$$:= F_\varepsilon^{(i)} + \varepsilon^m \widetilde{F}_k^{(i)}, \qquad j = i+1 \bmod 2 \qquad \text{(A.24)}$$

and the collision integrals computed with $R^{(i)}$ replaced by $R_{k-1}^{(i)}$; moreover, we put $R_0^{(i)} = 0$, $i = 1, 2$. Lemma A.2 and an inductive argument assure that the sequences $R_k^{(i)}$ are uniformly bounded for ε sufficiently small. In fact, setting $\bar{\alpha} = \alpha_{F_\varepsilon^{(i)}}$, we have $\alpha_k = \alpha_{F_k^{(i)}} \leqslant \bar{\alpha} + C\varepsilon^m \|R_{k-1}\|_1$. By a standard argument $\|R_{k-1}\|_1$ is bounded by a constant $\lambda(\alpha_{k-1})$ with $\lambda(\cdot)$ some monotone function. Hence, $\alpha = \sup \alpha_k$ satisfies the inequality $\alpha \leqslant \bar{\alpha} + \varepsilon^m \lambda(\alpha)$. By setting $\varepsilon^m \lambda(2\bar{\alpha}) \leqslant \bar{\alpha}$, we conclude $\alpha < 2\bar{\alpha}$ and hence the uniform boundedness of the sequences $R_k^{(i)}$ for ε sufficiently small.

The differences $\delta R_k^{(i)} := R_k^{(i)} - R_{k-1}^{(i)}$ can be decomposed again in high and low velocity parts δg_k, δh_k, with $\delta g_k = (\delta g_k^{(1)}, \delta g_k^{(2)})$ and $\delta h_k = (\delta h_k^{(1)}, \delta h_k^{(2)})$ solve the equations

$$D_t \, \delta g_k + (\varepsilon^m \tilde{F}_{k-1} + F_\varepsilon) \cdot \nabla_v \, \delta g_k = N(\delta g_k, \delta h_k)$$
$$D_t \, \delta h_k + (\varepsilon^m \tilde{F}_{k-1} + F_\varepsilon) \cdot \nabla_v \, \delta h_k = N'(\delta g_k, \delta h_k)$$

where $N = (N^{(1)}, N^{(2)})$, $N' = (N'^{(1)}, N'^{(2)})$ and $N^{(1)}$, $N'^{(1)}$ are the r.h.s. in (A.11) with

$$D^{(i)} = -\varepsilon \, \delta \tilde{F}_k \cdot \nabla_v (g_{k-1} + h_{k-1}) + [J(R_{k-1}^{(i)}, R_{k-1}^{(i)}) - J(R_{k-2}^{(i)}, R_{k-2}^{(i)})]$$
$$+ [J(R_{k-1}^{(i)}, R_{k-1}^{(j)}) - J(R_{k-2}^{(i)}, R_{k-2}^{(j)})]$$

with $\delta \tilde{F}_k^{(i)} := (\tilde{F}_k^{(i)} - \tilde{F}_{k-1}^{(i)}) = \mathbf{K} \circledast [R_{k-1}^{(j)} - R_{k-2}^{(j)}]$. By Lemma A.2 the solutions satisfy (A.20) and (A.21), so that

$$\|\delta g_k\|_{j,s} \leqslant C\varepsilon^{m-s+1} \|\tilde{v}^{-1}[\delta \tilde{F}_k \cdot \nabla_v (g_{k-1} + h_{k-1})]\|_{j+2,s} \tag{A.25}$$

$$\|\delta h_k\|_{j,s} \leqslant \varepsilon^{m-s+2} \|\tilde{v}^{-1}[\delta \tilde{F}_k \cdot \nabla_v (g_{k-1} + h_{k-1})]\|_{j,s} \tag{A.26}$$

We have

$$\|\delta \tilde{F}_k \cdot \nabla_v (g_{k-1} + h_{k-1})\|_{j0}^2$$
$$\leqslant C \sup_{x,v} (1+|v|^2)^j \, |\nabla_v (\delta g_k + \delta h_k)| \sup_v \int dx \, |\delta \tilde{F}_k|^2$$
$$\leqslant C \int dx \left| \int dv \int dy \, K(x-y)(R_{k-1} - R_{k-2}) \right|^2$$
$$\leqslant C \int dx \left| \sup_v (1+|v|^2)^j \int dy \, K(x-y)(\delta g_{k-1} + \delta h_{k-1}) \right|^2$$
$$\leqslant C \left| \left(\int dy \, |K(x-y)|^2 \right)^{1/2} \sup_v (1+|v|^2)^j \left(\int dy \, |\delta g_{k-1} + \delta h_{k-1}|^2 \right)^{1/2} \right|^2$$
$$\leqslant C[\|\delta g_{k-1}\|_{j0}^2 + \|\delta h_{k-1}\|_{j0}^2]$$

To get the second inequality we have used that the norms $\|\nabla_v \delta g_k\|_{j,s}$ and $\|\nabla_v \delta h_k\|_{j,s}$ are finite by Lemma A.2, so that the supremum over x in the first row exists finite. The last inequality is a consequence of the fact that the Kac potential is bounded and that the space integration is on a torus. Finally, by (A.25) and (A.26) we have

$$\|\delta g_k\|_{j,0} \leqslant c\varepsilon^{m-s+1} \|\delta g_{k-1}\|_{j,s} + \|\delta h_{k-1}\|_{j,0}]$$

$$\|\delta h_k\|_{j,0} \leqslant c\varepsilon^{m-s+1} \|\delta g_{k-1}\|_{j,s} + \|\delta h_{k-1}\|_{j,0}]$$

We remark that it is possible to prove that the norm $\|\cdot\|_{j,z}$ for the remainders $g^{(i)}$, $h^{(i)}$ are bounded with $z = s-(d-1)$ and $s < m$. ∎

APPENDIX B

In this Appendix we show how to bound the remainders which are solutions of (6.7). The method we use is different from the one in Appendix A. In fact in this case the lowest order is a global Maxwellian and we do not need to introduce the decomposition into low and high velocity. Also in this case we need a Theorem on the regularity of the terms of the expansion analogous to Theorem A.1.

Theorem B.1. Given $\varepsilon > 0$, assume that there exists a sufficiently smooth solution of the incompressible Navier–Stokes equations (5.28) in $(0, t_0]$. Then there is a constant $c > 0$ and s depending on the smoothness of the solution of INS such that the terms in the expansion f_i, ϕ_i, $i = 2,..., K$ solutions of the Eqs. (6.5) and (6.6) satisfy

$$\|f_i\|_{j,s} \leqslant c, \qquad \|\phi_i\|_{j,2} \leqslant c \tag{B.1}$$

$$\|\partial_v f_i\|_{j,s} \leqslant c, \qquad \|\partial_v \phi_i\|_{j,2} \leqslant c \tag{B.2}$$

for any j.

We write (6.7) for the variables $R^{(i)}$, $1 = 1, 2$ defined in (A.3)

$$\begin{aligned}
\partial_t R^{(1)} &+ \varepsilon^{-1}[v \cdot R^{(1)} + F^{(1)} \cdot \nabla_v R^{(1)}] \\
&= \varepsilon^{-2}[\mathscr{L} R^{(1)} + \Gamma R^{(1)} + \Theta R^{(2)}] \\
&\quad + \varepsilon^{-1}[\mathscr{L}_1 R^{(1)} + \Gamma_2 R^{(1)} + \Theta_1 R^{(2)}] + [\mathscr{L}'_1 R^{(1)} + \Gamma'_2 R^{(1)} + \Theta'_1 R^{(2)}] \\
&\quad + \varepsilon^{m-2}[J(R^{(1)}, R^{(1)}) + J(R^{(1)}, R^{(2)}) + A^{(1)}]
\end{aligned}$$

$$\begin{aligned}\partial_t R^{(2)} + \varepsilon^{-1}[v \cdot R^{(2)} + F^{(2)} \cdot \nabla_v R^{(2)}] \\ = \varepsilon^{-2}[\mathscr{L} R^{(2)} + \Gamma R^{(2)} + \Theta R^{(1)}] \\ + \varepsilon^{-1}[\mathscr{L}_2 R^{(2)} + \Gamma_1 R^{(2)} + \Theta_2 R^{(1)}] + [\mathscr{L}'_2 R^{(2)} + \Gamma'_1 R^{(2)} + \Theta'_2 R^{(1)}] \\ + \varepsilon^{m-2}[J(R^{(2)}, R^{(2)}) + J(R^{(2)}, R^{(1)}) + A^{(2)}]\end{aligned} \tag{B.3}$$

Θ, $\mathscr{L}$, Γ are defined as $\bar{\Theta}$, $\bar{\mathscr{L}}$, $\bar{\Gamma}$ in the list after (A.4) after substituting the global Maxwellian $M_0(\bar{\rho}, \bar{T})$ to $\bar{M}$. Finally,

$$\mathscr{L}_i g = J(f_1^{(i)}, g) + J(g, f_1^{(i)})], \qquad \Theta_i g = J(\varphi_1^{(i)}, g), \qquad \Gamma_i g = J(g, f_1^{(i)})$$

$$\mathscr{L}'_i g = \sum_{h=2}^{K} \varepsilon^{h-2}[J(f_h^{(i)}, g) + J(g, f_h^{(i)})]$$

$$\Theta'_i g = J\left(\sum_{n=2}^{K} \varepsilon^{n-2} \varphi_n^{(i)}, g\right), \qquad \Gamma'_i g = \sum_{h=2}^{K} \varepsilon^{h-2} J(g, f_h^{(i)})$$

The first step is to consider the linear problem associated to (B.3), namely to study (B.3) with the last terms $D^{(i)} := J(R^{(i)}, R^{(i)}) + J(R^{(i)}, R^{(j)}) + A^{(i)}$, $i = 1, 2$, $i \neq j$ given and $F^{(1)}$, $F^{(2)}$ fixed, independent of $R^{(1)}$, $R^{(2)}$, Moreover remembering that the forces vanish to the lowest order in ε, we assume that the L_∞ norms of $F^{(i)}$ and their gradients are bounded by some constant $\varepsilon\alpha_F$. The role of the constant α_F is similar to the one discussed in the previous appendix and we do not repeat the iterative argument in this case. We will just provide an estimate for the L^2 norm in (x, v) $\|R\|_2 := \|R^{(1)}\|_2 + \|R^{(2)}\|_2$ for the solution $R = (R^{(1)}, R^{(2)})$ of this problem, the rest of the argument being standard (see for example [ELM98]). We put $R^{(i)} = \sqrt{M_0}\, \Psi^{(i)}$ so that

$$\begin{aligned}\partial_t \Psi^{(1)} + \varepsilon^{-1}\left[v \cdot \nabla \Psi^{(1)} + F^{(1)} \cdot \nabla_v \Psi^{(1)} - \frac{1}{2} \Psi^{(1)} F^{(1)} \cdot v\right] \\ = \varepsilon^{-2}[L\Psi^{(1)} + G\Psi^{(1)} + T\Psi^{(2)}] + \varepsilon^{-1}[L_1 R^{(1)} + G_2 \Psi^{(1)} + T_1 \Psi^{(2)}] \\ + [L'_1 \Psi^{(1)} + G'_2 \Psi^{(1)} + T'_1 \Psi^{(2)}] + \varepsilon^{m-2} \frac{D^{(1)}}{\sqrt{M_0}}\end{aligned} \tag{B.4}$$

where the relation between the old operators $\mathscr{L}$, Θ, Γ, $\mathscr{L}_i$, Θ_i, Γ_i, $\mathscr{L}'_i$, Θ'_i, Γ'_i and the new ones L, T, G, L_i, T_i, G_i, L'_i, T'_i, G'_i is of the form

$$\mathscr{L} f = \frac{1}{\sqrt{M_0}} L \sqrt{M_0}\, f$$

It is easy to see that, setting $\|\Psi\|_{L_2}^2 = \|\Psi^{(1)}\|_{L_2}^2 + \|\Psi^{(2)}\|_{L_2}^2$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Psi\|_{L_2}^2 = \varepsilon^{-2}\Big[& \langle \Psi^{(1)}, L\Psi^{(1)}\rangle + \langle \Psi^{(2)}, L\Psi^{(2)}\rangle \\
& -\frac{1}{2}\varepsilon \sum_{i=1}^{2} F^{(i)} \cdot \langle \Psi^{(i)}, v\Psi^{(i)}\rangle \Big] \\
& + \varepsilon^{-2}[\langle \Psi^{(1)}, M_0^{-1/2} J(M_0, \sqrt{M_0}\, \Psi^{(2)})\rangle \\
& + \langle \Psi^{(1)}, M_0^{-1/2} J(\sqrt{M_0}\, \Psi^{(1)}, M_0)\rangle \\
& + \langle \Psi^{(2)}, M_0^{-1/2} J(M_0, \sqrt{M_0}\, \Psi^{(1)})\rangle \\
& + \langle \Psi^{(2)}, M_0^{-1/2} J(\sqrt{M_0}\, \Psi^{(2)}, M_0)\rangle] \\
& + \varepsilon^{-1}[\langle \Psi^{(1)}, L_1 \Psi^{(1)}\rangle + \langle \Psi^{(2)}, L_2 \Psi^{(2)}\rangle \\
& + \langle \Psi^{(1)}, (G_2 \Psi^{(1)} + T_1 \Psi^{(2)})\rangle \\
& + \langle \Psi^{(2)}, (G_1 \Psi^{(2)} + T_2 \Psi^{(1)})\rangle] + [\langle \Psi^{(1)}, L_1' \Psi^{(1)}\rangle \\
& + \langle \Psi^{(2)}, L_2' \Psi^{(2)}\rangle + \langle \Psi^{(1)}, (G_2' \Psi^{(1)} + T_1' \Psi^{(2)})\rangle \\
& + \langle \Psi^{(2)}, (G_1' \Psi^{(2)} + T_2' \Psi^{(1)})\rangle] \\
& + \varepsilon^{m-2} \sum_{i=1}^{2} \left\langle \Psi^{(i)}, \frac{D^{(i)}}{\sqrt{M_0}} \right\rangle
\end{aligned} \tag{B.5}$$

First of all, we observe that the terms in the second square bracket are non positive [CC]. To estimate the other terms we will use the strict negativity of the operator L (see (4.20)) on the space orthogonal to the collision invariants and the following estimate on the operator $J(f, h)$ (see for example [GPS]): for any Maxwellian M and for any $y \in [-1, 1]$

$$\int_{\mathbb{R}^3} dv \frac{|J(\sqrt{M}\, f, \sqrt{M}\, h)|^2}{\nu M} \leqslant \int_{\mathbb{R}^3} dv\, \nu\, |f|^2 \int_{\mathbb{R}^3} dv\, \nu\, |h|^2 \tag{B.6}$$

This inequality and the bounds on the f_n's imply the following bounds:

$$\left| \left\langle \sum_{i=1}^{2} \Psi^{(i)} L_i \Psi^{(i)} \right\rangle \right| \leqslant C \left\| \sum_{i=1}^{2} \sqrt{\nu}\, \bar{\Psi}^{(i)} \right\|_2 \|\Psi^{(i)}\|_{L_2} \|M_0^{-1/2} f_1^{(i)}\|_{j,s} \tag{B.7}$$

$$\left| \left\langle \sum_{i=1}^{2} \Psi^{(i)} L_i' \Psi^{(i)} \right\rangle \right| \leqslant C \|\sqrt{\nu}\, \bar{\Psi}\|_{L_2} \|\Psi\|_{L_2} \left\| M_0^{-1/2} \sum_{n=2}^{7} f_n \right\|_{j,s} \tag{B.8}$$

where $\bar{g}$ denotes the projection of a function g on the orthogonal to the invariant space of L, while $\hat{g}$ is the projection on the invariant space. Note that the presence of the product $\|\sqrt{v}\,\bar{\Psi}\|_2\,\|\Psi\|_2$ depends on the fact that L_i and L_i' are both orthogonal to the collision invariants.

Similar estimates hold for the terms involving the other operators. By using the bounds on the f_n's and ϕ_n's and after some algebra, the terms in the forth, fifth and sixth rows are bounded by

$$C\,\|\sqrt{v}\,\bar{\Psi}\|_{L_2}\,\|\Psi\|_{L_2}$$

To bound the last term in the first square bracket of (B.5), we note that

$$\varepsilon^{-1}\left|\sum_{i=1}^{2} F^{(i)}\cdot\langle \Psi^{(i)}, v\Psi^{(i)}\rangle\right| \leqslant \alpha_F[\,\|\sqrt{v}\,\bar{\Psi}\|_{L_2}^2 + C\,\|\hat{\Psi}\|_{L_2}^2$$

and we assume ε so small that $\varepsilon^2\alpha_F < 1/2$.

We integrate (B.5) in time between 0 and t_0. With the notation $\Psi_t(\cdot) = \Psi(\cdot, t)$, we get

$$\tfrac{1}{2}\,\|\Psi_{t_0}\|_{L_2}^2 \leqslant C\int_0^{t_0} dt\{-\varepsilon^{-2}[\tfrac{1}{2}\,\|\sqrt{v}\,\bar{\Psi}_t\|_{L_2}^2 + C_F\,\|\sqrt{v}\,\Psi_t\|_{L_2}^2]$$
$$+\,C(\varepsilon^{-1}+1)\,\|\sqrt{v}\,\bar{\Psi}_t\|_{L_2}\,\|\Psi_t\|_{L_2} + \varepsilon^{m-2}\,\|D(\cdot, t)\|_{L_2}^2\} \qquad \text{(B.9)}$$

The first term in the second line is due to the bounds (B.7) and (B.8). Moreover $\|M_0^{-1/2} f_1\|_{j,s}$ and $\|M_0^{-1/2} f_1\|_{j,s}$ are bounded by the regularity of the solutions of the macroscopic equations for $0 < t < T_0$ and $\|M_0^{-1/2}\sum_{n=2}^{K} f_n\|_{j,s} \leqslant C$ by Theorem B.1.

Using the inequality

$$-\frac{1}{\varepsilon^2}\,x^2 + (c_1\varepsilon^{-1} + c_2)\,xy \leqslant (c_1 + c_2\varepsilon)^2\;y^2/4$$

valid for any positive ε, x, y with $x = \|\sqrt{v}\,\bar{\Psi}\|_2$, $y = \|\Psi\|$ and suitable constants c_1 and c_2, we get (since $\Psi(\cdot, 0) = 0$)

$$\|\Psi(\cdot, t_0)\|_{L_2}^2 \leqslant \int_0^{t_0} dt\; C_F[\,\|\Psi(\cdot, t)\|_{L_2}^2 + \|M_0^{-1/2} D(\cdot, t)\|_{L_2}^2]$$

In conclusion, by the use of the Gronwall lemma, for ε sufficiently small, we get:

$$\sup_{0\leqslant t\leqslant t_0}\;\|\Psi(\cdot, t)\|_{L_2} \leqslant C(t_0, \alpha_F)\sup_{t\in(0, t_0]}\;\|M_0^{-1/2} D(\cdot, t)\|_{L_2} \qquad \text{(B.10)}$$

The bounds for the Sobolev norm of higher order in x, v are obtained by studying the equations for the derivatives as explained in Appendix A. Finally, writing the equations for the remainders in the integral form and using the property of the linearized Boltzmann operators of improving the regularity in v, we get the analogous of Lemma A.2.

Lemma B.2. There is an $\varepsilon_0 > 0$ such that any solution to the linear problem (B.4), with D and F_i given, after choosing $K = 2m$ satisfies for $j > 3$, $s < s_0$ and any $0 < \varepsilon \leqslant \varepsilon_0$

$$\|g^{(i)}\|_{j,s} + \|\partial_v g^{(i)}\|_{j,s-1} \leqslant \varepsilon^{m-2} C_F[\|D^{(i)}\|_{j+2,s} + \|\tilde{v}^{-1}\partial_v D^{(i)}\|_{j+2,s}] \tag{B.11}$$

$$\|h^{(i)}\|_{j,s} + \|\partial_v h^{(i)}\|_{j,s-1} \leqslant \varepsilon^{m-2} C_F[\|D^{(i)}\|_{j,s} + \|\tilde{v}^{-1}\partial_v D^{(i)}\|_{j,s}] \tag{B.12}$$

The dependence on the force in the bounds (B.11), (B.12) does not affect the argument given in Appendix A to solve the non-linear problem, because in the bounds for R_k the constant C_{F_k} will depend on the norm of R_{k-1}.

APPENDIX C

To show formally the convergence of the microscopic one particle distribution functions to the solution of the VBE in the Grad–Boltzmann limit, let us consider the hierarchy for the rescaled correlation, functions r_{j_r, j_b} of j_r particles of species r and j_b of species b, defined as

$$\begin{aligned} &r_{j_r, j_b}(z_1^r,..., z_{j_r}^r; z_1^b,..., z_{j_b}^b; \tau) \\ &\quad = \delta^{-(j_r + j_b)} \frac{N_r!}{(N_r - j_r)!} \frac{N_b!}{(N_b - j_b)!} \\ &\quad \times \int_{(\Lambda \times \mathbb{R}^3)^{(N - j_r - j_b)}} dz_{j_r+1}^r \cdots dz_{N_r}^r \, dz_{j_b+1}^b \cdots dz_{N_b}^b \\ &\quad \times \mu_N(\delta^{-1}q_1^r, v_1^r \cdots, \delta^{-1}q_{j_r}^r, v_{j_r}^r; \delta^{-1}q_1^b, v_1^b \cdots, \delta^{-1}q_{j_b}^b; \delta^{-1}\tau)) \end{aligned} \tag{C.1}$$

where $z^\alpha = (q^\alpha, v^\alpha)$ is the phase space point of a particle of species α and μ_N is the probability distribution solution of the Liouville equation

$$\partial_{\tau_m}\mu_N + \sum_{i=1}^{N} v_i \cdot \nabla_{\xi_i}\mu_N - A_\ell \sum_{\alpha \neq \beta} \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \nabla_{\xi_i^\alpha} U_\ell(|\xi_i^\alpha - \xi_j^\beta|) \cdot \nabla_{v_i^\alpha}\mu_N = 0$$

which is valid in Γ_N, i.e., where the hard spheres do not overlap. On the boundary of Γ_N we assume the boundary conditions

$$\mu_N(\xi_1, v_1, \dots, \xi_N, v_N; \tau_m) = \mu_N(\xi_1, v_1, \dots, \xi_i, v'_i, \dots, \xi_j, v'_j, \dots, \xi_N, v_N; \tau_m)$$

if

$$|\xi_i - \xi_j| = 1, \qquad i \neq j$$

where $v'_i = v_i - \omega[\omega \cdot (v_i - v_j)]$, $v'_j = v_j + \omega[\omega \cdot (v_i - v_j)]$ with ω the unit vector directed as $\xi_i - \xi_j$. The above conditions merely state the conservation of the probability during an elastic collision. As pointed out before, contacts of more than two particles have null Lebesgue measure, so they do not affect above definition.

The rescaled correlation functions satisfy a hierarchy of equations of the form

$$\begin{aligned} \partial_\tau r_{j_r, j_b} &+ \sum_{i=1}^{j_r} \left[v_i^r \cdot \nabla_{q_i^r} r_{j_r, j_b} + \delta^3 \sum_{j=1}^{j_b} \nabla_{q_i^r} V_\gamma(|q_i^r - q_j^b|) \cdot \nabla_{v_i^r} r_{j_r, j_b} \right] \\ &+ \sum_{i=1}^{j_b} \left[v_i^b \cdot \nabla_{q_i^b} r_{j_r, j_b} + \delta^3 \sum_{j=1}^{j_r} \nabla_{q_i^b} V_\gamma(|q_i^b - q_j^r|) \cdot \nabla_{v_i^b} r_{j_r, j_b} \right] \\ &= \sum_{i=1}^{j_r} \left[\mathscr{B}_{\delta, i}^{r, r} r_{j_r+1, j_b} + \mathscr{B}_{\delta, i}^{b, r} r_{j_r, j_b+1} + \mathscr{V}_i^{b, r} r_{j_r, j_b+1} \right] \\ &\quad + \sum_{i=1}^{j_b} \left[\mathscr{B}_{\delta, i}^{b, b} r_{j_r, j_b+1} + \mathscr{B}_{\delta, i}^{r, b} r_{j_r+1, j_b} + \mathscr{V}_i^{r, b} r_{j_r+1, j_b} \right] \end{aligned}$$

where, with the notation $\underline{z}_k = (z_1, \dots, z_k)$, we have

$$\begin{aligned} &(\mathscr{B}_{\delta, i}^{r, r} r_{j_r+1, j_b})(\underline{z}_{j_r}^r; \underline{z}_{j_b}^b) \\ &\quad = \int_{\mathbb{R}^3} dv_{j_r+1}^r \int_{S_+^2} d\omega (v_{j_r+1}^r - v_i^r) \cdot \omega [r_{j_r+1, j_b}((\underline{z}_{j_r+1}^r)'; \underline{z}_{j_b}^b) \\ &\qquad - r_{j_r+1, j_b}(\overline{\underline{z}_{j_r+1}^r}; \underline{z}_{j_b}^b)] \end{aligned}$$

with

$$\begin{aligned} S_+^2 &= \{\omega \in \mathbb{R}^3 \mid |\omega| = 1, \omega \cdot (v_{j_r+1}^r - v_i^r) > 0\} \\ (\underline{z}_{j_r+1}^r)' &= (z_1^r, \dots (z_i^r)', \dots, z_{j_r}^r, (z_{j_r+1}^r)') \\ \overline{\underline{z}_{j_r+1}^r} &= (z_1^r, \dots, \bar{z}_i^r, \dots, z_{j_r}^r, \bar{z}_{j_r+1}^r) \end{aligned}$$

and, for any z, z_*, the phase points z', z'_*, $\bar{z}$ and $\bar{z}_*$ are defined by

$$q' = q, \qquad q'_* = q + \delta\omega, \qquad v' = v - \omega(\omega \cdot (v - v_*))$$

$$v'_* = v_* + \omega(\omega \cdot (v - v_*)), \qquad \bar{q} = q, \qquad \bar{q}_* = q - \delta\omega, \qquad \bar{v} = v, \qquad \bar{v}_* = v_*$$

Moreover,

$$\begin{aligned}
&(\mathscr{B}^{b,r}_{\delta,i} r_{j_r, j_b+1})(\underline{z}^r_{j_r}; \underline{z}^b_{j_b}) \\
&\quad = \int_{\mathbb{R}^3} dv^b_{j_r+1} \int_{S^2_+} d\omega (v^b_{j_b+1} - v^r_i) \cdot \omega [r_{j_r, j_b+1}((\underline{z}^r_{j_r})'; (\underline{z}^b_{j_b+1})') \\
&\qquad - r_{j_r, j_b+1}(\overline{\underline{z}^r_{j_r}}; \overline{\underline{z}^b_{j_b+1}})]
\end{aligned}$$

where

$$(\underline{z}^r_{j_r})' = (z^r_1, \ldots (z^r_i)', \ldots, z^r_{j_r}), \qquad (\underline{z}^b_{j_b+1})' = (z^b_1, \ldots, z^b_{j_b}, (z^b_{j_b+1})')$$

$$\overline{\underline{z}^r_{j_r}} = (z^r_1, \ldots \bar{z}^r_i, \ldots, z^r_{j_r}), \qquad \overline{\underline{z}^b_{j_b+1}} = (z^b_1, \ldots, z^b_{j_b}, \bar{z}^b_{j_b+1})$$

The collision terms $\mathscr{B}^{b,b}_{\delta,i}$ and $\mathscr{B}^{r,b}_{\delta,i}$ are defined in a similar way. Furthermore

$$\begin{aligned}
&(\mathscr{V}^{b,r}_i r_{j_r, j_b+1})(\underline{z}^r_{j_r}; \underline{z}^b_{j_b}) \\
&\quad = -\int_{\mathbb{R}^3} dv^b_{j_r+1} \int_{\Lambda} dq^b_{j_b+1} \gamma^3 \nabla_{q^r_i} U_\gamma(|q^r_i - q^b_{j_b+1}|) \nabla_{v^r_i} r_{j_r, j_b+1}(z^r_{j_r}; \underline{z}^b_{j_b+1})
\end{aligned}$$

and a similar expression for $\mathscr{V}^{r,b}_i$. Taking formally the limit $\delta \to 0$, the limiting correlations satisfy the following Vlasov–Boltzmann hierarchy:

$$\begin{aligned}
&\partial_\tau r_{j_r, j_b} + \sum_{i=1}^{j_r} v^r_i \cdot \nabla_{q^r_i} r_{j_r, j_b} + \sum_{i=1}^{j_b} v^b_i \cdot \nabla_{q^b_i} r_{j_b, j_b} \\
&\quad = \sum_{i=1}^{j_r} [\mathscr{B}^{r,r}_i r_{j_r+1, j_b} + \mathscr{B}^{b,r}_i r_{j_r, j_b+1} + \mathscr{V}^{b,r}_i r_{j_r, j_b+1}] \\
&\qquad + \sum_{i=1}^{j_b} [\mathscr{B}^{b,b}_i r_{j_r, j_b+1} + \mathscr{B}^{r,b}_i r_{j_r+1, j_b} + \mathscr{V}^{r,b}_i r_{j_r+1, j_b}] \qquad \text{(C.3)}
\end{aligned}$$

where,

$$
\begin{aligned}
(\mathscr{B}_i^{r,r} r_{j_r+1, j_b})(\underline{z}_{j_r}^r; \underline{z}_{j_b}^b) &= \int_{\mathbb{R}^3} dv_{j_r+1}^r \int_{S_+^2} d\omega (v_{j_r+1}^r - v_i^r) \cdot \omega [r_{j_r+1, j_b}(\underline{z}_{j_r+1}^r)'; \underline{z}_{j_b}^b) \\
&\quad - r_{j_r+1, j_b}(\underline{z}_{j_r+1}^r; \underline{z}_{j_b}^b)]
\end{aligned}
$$

and, for any z, z_*, the phase points z', z'_* are defined by

$$
q' = q, \qquad q'_* = q, \qquad v' = v - \omega(\omega \cdot (v - v_*)), \qquad v'_* = v_* + \omega(\omega \cdot (v - v_*))
$$

Similar modifications provide the other terms of the Vlasov–Boltzmann hierarchy.

It is easy to see that if the initial condition of *molecular chaos*

$$
r_{j_r, j_b}(\underline{z}_{j_r}^r; \underline{z}_{j_b}^b; 0) = \prod_{i=1}^{j_r} f^r(z_i^r, 0) \prod_{k=1}^{j_b} f^b(z_k^b, 0)
$$

is satisfied, then the correlation functions stay factorized at positive times τ and $f^r(q, v, \tau)$ and $f^b(q, v, \tau)$ are the solutions of the coupled Vlasov–Boltzmann equations

$$
\begin{aligned}
\partial_\tau f^r(q, v, \tau) + v \cdot \nabla_q f^r(q, v, \tau) + F^r \cdot \nabla_v f^r(q, v, \tau) &= J(f^r, f^r + f^b) \\
\partial_\tau f^b(q, v, \tau) + v \cdot \nabla_q f^b(q, v, \tau) + F^b \cdot \nabla_v f^b(q, v, \tau) &= J(f^b, f^r + f^b)
\end{aligned}
\tag{C.4}
$$

where

$$
F^r(q, \tau) = -\nabla_q \int_\Omega dq' \, \gamma^3 (\nabla U_\gamma)(|q - q'|) \int_{\mathbb{R}^3} dv \, f^b(q', v, \tau) \tag{C.5}
$$

$$
F^b(q, \tau) = -\nabla_q \int_\Omega dq' \, \gamma^3 (\nabla U_\gamma)(|q - q'|) \int_{\mathbb{R}^3} dv \, f^r(q', v, \tau) \tag{C.6}
$$

and

$$
J(f, g) = \int_{\mathbb{R}^3} dv_* \int_{S_+^2} d\omega (v - v_*) \cdot \omega [f(v') \, g(v'_*) - f(v) \, g(v_*)] \tag{C.7}
$$

Summarizing, we obtained formally the Vlasov–Boltzmann equations for a binary mixture, where the Boltzmann collision kernel terms are due to the short range interaction while the Vlasov self consistent force is due to the repulsive weak long range interaction. If, instead of the hard core interaction the short range force is given by a finite range potential, we would get formally the same equations but with a different cross section.

We want to stress that an important step is missing in order to make the above derivation rigorous. The first rigorous result on the derivation of the Boltzmann equation has been given by Lanford [Lan] where the convergence of the correlation functions is proven in L_∞-norms. On the other hand, the derivation of the Vlasov equation is based on the use of the variation norm and we have not been able to find a norm suited for both terms. The only related result, as far as we know, has been obtained in [GM] and is about a stochastic particle systems converging to a Vlasov–Boltzmann equation with a modified Boltzmann kernel (Povzner). The proof is based on martingale methods. In the linear case of a Lorentz gas with a Kac potential term it is possible to prove the convergence to a Boltzmann equation with a linear collision term and a non-linear self-consistent force term [MR].

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